

Group measure space decomposition of II_1 factors and W^* -superrigidity

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Abstract

We prove a “unique crossed product decomposition” result for group measure space II_1 factors $L^\infty(X) \rtimes \Gamma$ arising from arbitrary free ergodic probability measure preserving (p.m.p.) actions of groups Γ in a fairly large family \mathcal{G} , which contains all free products of a Kazhdan group and a non-trivial group, as well as certain amalgamated free products over an amenable subgroup. We deduce that if T_n denotes the group of upper triangular matrices in $\text{PSL}(n, \mathbb{Z})$, then any free, mixing p.m.p. action of $\Gamma = \text{PSL}(n, \mathbb{Z}) *_{T_n} \text{PSL}(n, \mathbb{Z})$ is W^* -superrigid, i.e. any isomorphism between $L^\infty(X) \rtimes \Gamma$ and an arbitrary group measure space factor $L^\infty(Y) \rtimes \Lambda$, comes from a conjugacy of the actions. We also prove that for many groups Γ in the family \mathcal{G} , the Bernoulli actions of Γ are W^* -superrigid.

1 Introduction

Rigidity results have by now appeared in many areas of mathematics, and in several forms. The most frequently encountered is when two mathematical objects with rich structure that are known to be equivalent in some “weak sense”, which ignores part of the structure, are shown to be isomorphic as objects with the full structure. In the best of cases, such a result will also show that morphisms which are equivalences in the weak sense are equivalent to isomorphisms in the stronger category, thus leading to complete classification results and calculation of invariants.

Von Neumann algebras (also called W^* -algebras) provide a most natural framework for rigidity. In fact, such phenomena are at the very core of this subject, relating it at the outset with group theory and ergodic theory. This is due to the Murray and von Neumann classical *group measure space construction*, which associates to a free ergodic measure preserving action $\Gamma \curvearrowright X$, of a countable group Γ on a probability space (X, μ) , a von Neumann algebra (called II_1 factor) $L^\infty(X) \rtimes \Gamma$, through a crossed product type construction [MvN36]. The study of these objects in terms of their “initial data” $\Gamma \curvearrowright X$ has been a central theme of the subject since the early 1940s. It soon led to a new area in ergodic theory, studying group actions up to *orbit equivalence* (OE), i.e. up to isomorphism of probability spaces carrying the orbits of actions onto each other, since an OE of actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ has been shown in [Si55] to be “the same as” an algebra isomorphism $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ taking the *group measure space Cartan subalgebras* $L^\infty(X)$, $L^\infty(Y)$ onto each other.

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Thus, W^* -equivalence (or *von Neumann equivalence*) of group actions, requiring isomorphism of their group measure space algebras, is weaker than OE. Since there are examples of non-OE actions whose group measure space factors are isomorphic [CJ82, OP08], it is in general strictly weaker. In turn, it has been known since [MvN43, Dy59] that OE is much weaker than classical *conjugacy* (or *isomorphism*), which for free actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ requires isomorphism of probability spaces $\Delta : (X, \mu) \simeq (Y, \nu)$ satisfying $\Delta \Gamma \Delta^{-1} = \Lambda$ (so in particular $\Gamma \simeq \Lambda$). Rigidity in this context occurs whenever one can establish that W^* - or OE-equivalence of certain group actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$, forces the groups, or the actions, to share some common properties. The ideal such result, labeled W^* - (respectively OE-) *superrigidity*, recovers the isomorphism class of $\Gamma \curvearrowright X$, from its W^* -class (resp. OE-class).

W^* - and OE-rigidity can only occur for non-amenable groups, since by classical results of Connes [Co76], all II_1 factors $L^\infty(X) \rtimes \Gamma$ with Γ amenable are mutually isomorphic and by [OW80, CFW81] they are undistinguishable under OE as well. But the non-amenable case is extremely complex and although signs of rigidity were detected early on [MvN43, Dy63, McD70, Co75], for many years progress has been slow, despite several breakthrough discoveries in the 1980s [Co80a, Zi80, CJ85, CH89]. This changed dramatically over the last decade, with the advent of a variety of striking rigidity results, in both group measure space II_1 factors and OE ergodic theory: [Fu99a, Fu99b, Ga00, Ga01, Po01, MS02, Hj02, Oz03, Po03, Po04, Po05, HK05, IPP05, Po06a, PV06, Pe06, Ki06, Va07, OP07, Io08, OP08, PV08a, PV08b, PV08c, CH08, Ki09].

Our purpose in this paper is to investigate the most “extreme” of the W^* -rigidity phenomena mentioned above, i.e. W^* -superrigidity. Thus, we seek to find classes of group actions $\Gamma \curvearrowright X$ with the property that any isomorphism between $L^\infty(X) \rtimes \Gamma$ and any other group measure space factor $L^\infty(Y) \rtimes \Lambda$, arising from an arbitrary free ergodic p.m.p. action $\Lambda \curvearrowright Y$, comes from a conjugacy of the actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$.

Note that W^* -superrigidity for an action $\Gamma \curvearrowright X$ is equivalent to the “sum” between its OE-superrigidity and the uniqueness, up to unitary conjugacy, of $L^\infty(X)$ as a group measure space Cartan subalgebra in $L^\infty(X) \rtimes \Gamma$. This makes W^* -superrigidity results extremely difficult to obtain, since each one of these problems is notoriously hard. But while several large families of OE-superrigid actions have been discovered over the last ten years [Fu99b, Po05, Po06a, Ki06, Io08, Ki09], unique Cartan decomposition proved to be much more challenging to establish, and the only existing results cover very particular group actions. Thus, a first such result, obtained by Ozawa and the first named author in [OP07], shows that given any profinite action $\Gamma \curvearrowright X$, of a product of free groups $\Gamma = \mathbb{F}_{n_1} \times \cdots \times \mathbb{F}_{n_k}$, with $k \geq 1$, $2 \leq n_i \leq \infty$, any Cartan subalgebra of $M = L^\infty(X) \rtimes \Gamma$ (i.e. any maximal abelian subalgebra whose normalizer generates M), is unitary conjugate to $L^\infty(X)$. A similar result, covering a more general class of groups Γ , was then proved in [OP08]. More recently, Peterson showed in [Pe09] that factors arising from profinite actions of non-trivial free products $\Gamma = \Gamma_1 * \Gamma_2$, with at least one of the Γ_i not having the Haagerup property, have unique group measure space Cartan subalgebra, up to unitary conjugacy. But so far, none of these group actions could be shown to be OE-superrigid. Nevertheless, an intricate combination of results in [Io08, OP08, Pe09] were used to prove the existence of virtually W^* -superrigid group actions $\Gamma \curvearrowright X$ in [Pe09], by a Baire category argument (following [Fu99a], *virtual* means that the ensuing conjugacy of $\Gamma \curvearrowright X$ and the target actions $\Lambda \curvearrowright Y$ is up to finite index subgroups of Γ, Λ).

In this paper, we establish a very general unique Cartan decomposition result, which allows us to obtain a wide range of W^* -superrigid group actions. Thus, we first prove the uniqueness, up to unitary conjugacy, of the group measure space Cartan subalgebra in the II_1 factor given by an *arbitrary* free ergodic p.m.p. action of any group Γ belonging to a large family \mathcal{G} of amalgamated

free product groups. By combining this with Kida's OE-superrigidity in [Ki09], we deduce that any free, mixing p.m.p. action of $\Gamma = \mathrm{PSL}(n, \mathbb{Z}) *_{T_n} \mathrm{PSL}(n, \mathbb{Z})$ is W^* -superrigid. In combination with [Po05, Po06a], we prove that for many groups Γ in the family \mathcal{G} , the Bernoulli actions of Γ are W^* -superrigid. In combination with Gaboriau's work [Ga00] on cost, we find new groups Γ for which all group measure space II_1 factors $L^\infty(X) \rtimes \Gamma$ have trivial fundamental group.

1.1 Statements of main results

More precisely, our family \mathcal{G} contains all non-trivial free products $\Gamma = \Gamma_1 * \Gamma_2$ with Γ_1 satisfying one of the following rigidity properties: Γ_1 contains a non-amenable subgroup with the relative property (T) or Γ_1 contains two non-amenable commuting subgroups. The family \mathcal{G} also contains certain amalgamated free products $\Gamma_1 *_\Sigma \Gamma_2$ over amenable subgroups Σ , see Definition 5.1.

Our results can be summarized as follows.

Theorem 1.1 (See Theorem 5.2). *Let Γ be a group in the family \mathcal{G} and $\Gamma \curvearrowright (X, \mu)$ an arbitrary free ergodic p.m.p. action. Denote $M = L^\infty(X) \rtimes \Gamma$. Whenever $\Lambda \curvearrowright (Y, \eta)$ is a free ergodic p.m.p. action such that $M = L^\infty(Y) \rtimes \Lambda$, there exists a unitary $u \in M$ such that $L^\infty(Y) = u L^\infty(X) u^*$.*

We mention that the most general version of the above theorem (see Theorem 5.2) allows to handle amplifications of the group measure space factors M as well.

Theorem 1.2 (See Theorem 6.2). *Let $n \geq 3$ and denote by T_n the subgroup of upper triangular matrices in $\mathrm{PSL}(n, \mathbb{Z})$. Put $\Gamma = \mathrm{PSL}(n, \mathbb{Z}) *_{T_n} \mathrm{PSL}(n, \mathbb{Z})$. Then, every free p.m.p. mixing action of Γ is W^* -superrigid.*

Whenever Γ is an infinite group and (X_0, μ_0) a non-trivial probability space, denote by $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ the Bernoulli action of Γ with base space (X_0, μ_0) , given by $(g \cdot x)_h = x_{g^{-1}h}$ for all $g, h \in \Gamma$ and $x \in X_0^\Gamma$.

Theorem 1.3 (See Theorem 6.7). *The Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ of all of the following groups is W^* -superrigid.*

- $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ with the following assumptions: Γ_1 has property (T), Σ is an infinite, amenable, proper normal subgroup of Γ_2 and there exist $g_1, \dots, g_k \in \Gamma_1$ such that $\bigcap_{i=1}^k g_i \Sigma g_i^{-1} = \{e\}$.
For instance, we can take $\Gamma = \mathrm{PSL}(n, \mathbb{Z}) *_\Sigma (\Sigma \times \Lambda)$, where $\Sigma < T_n$ is an infinite subgroup of the upper triangular matrices and Λ is an arbitrary non-trivial group.
- $\Gamma = (H \times H) *_\Sigma \Gamma_2$ where H is a finitely generated non-amenable group with trivial center, Σ is an infinite amenable subgroup of H that we embed diagonally in $H \times H$ and Σ is a proper normal subgroup of Γ_2 .

We in fact obtain a more general version of this result as Theorem 6.7, which covers generalized Bernoulli actions, Gaussian actions and certain co-induced actions (see Examples 6.8, 6.9 and 6.10).

Our methods also provide the following new examples of II_1 factors which cannot be written as group measure space factors.

Theorem 1.4. *Let $\Gamma \in \mathcal{G}$ and assume that Γ is ICC. Let $\Gamma \curvearrowright (X, \mu)$ be an arbitrary ergodic p.m.p. action and put $M = L^\infty(X) \rtimes \Gamma$. Then, M is a II_1 factor. If, for some $t > 0$, the II_1 factor M^t admits a group measure space decomposition, then the action $\Gamma \curvearrowright (X, \mu)$ must be free. Thus, $L(\Gamma)$ and all the factors of the form $L^\infty(X) \rtimes \Gamma$ corresponding to non-free actions $\Gamma \curvearrowright X$, do not admit a group measure space decomposition.*

1.2 Rigidity of bimodules

Another typical rigidity paradigm encountered in mathematics is when certain invariants of mathematical objects which are supposed to take values in a certain range, are shown to take values in a much smaller subset. Group measure space II_1 factors provide a natural framework for this type of rigidity as well, due to Murray and von Neumann's *continuous dimension* and a related invariant for II_1 factors M : the *fundamental group* $\mathcal{F}(M)$. This is defined as the set of ratios $\tau(p)/\tau(q) \in \mathbb{R}_+$, over all projections $p, q \in M$ with $pMp \simeq qMq$, where τ denotes the (unique) normalized trace (=dimension function) on M . Equivalently, $\mathcal{F}(M) = \{t > 0 \mid M^t \simeq M\}$. Thus, since the range of the dimension function is all $[0, 1]$, the group $\mathcal{F}(M)$ seems to always be equal to \mathbb{R}_+ . Supporting evidence comes from the case $M = L^\infty(X) \rtimes \Gamma$ with Γ amenable, when this is indeed the case (cf. [MvN43]). So it came as a striking surprise when Connes showed that all factors $L^\infty(X) \rtimes \Gamma$ with Γ an ICC Kazhdan group and $\Gamma \curvearrowright X$ ergodic, have countable fundamental group ([Co80a]).

The important progress in W^* -rigidity in recent years, led to the first actual computations of fundamental groups $\mathcal{F}(M)$ of group measure space factors $M = L^\infty(X) \rtimes \Gamma$: from the first such examples in [Po01], where $\mathcal{F}(M) = 1$, to examples of factors M with $\mathcal{F}(M)$ any prescribed countable subgroup of \mathbb{R}_+ [Po03] (see also [IPP05, Ho07]), and most recently examples with $\mathcal{F}(M)$ uncountable, yet different from \mathbb{R}_+ [PV08a, PV08c]. We mention in this respect that all Bernoulli actions $\Gamma \curvearrowright (X, \mu)$ appearing in Theorem 1.3 give rise to II_1 factors $L^\infty(X) \rtimes \Gamma$ with trivial fundamental group (see Remark 6.6). In the same spirit, Corollary 5.3 provides new examples of groups Γ such that $L^\infty(X) \rtimes \Gamma$ has trivial fundamental group for all free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$.

Another occurrence of the same type of rigidity paradigm is related to Jones' index for subfactors, a numerical invariant for inclusions of II_1 factors $N \subset M$ which, like the fundamental group, is defined with the help of the Murray-von Neumann continuous dimension. A priori, the range of the index could well be all \mathbb{R}_+ , but in his seminal work [Jo83], Jones proved that it is subject to very surprising restrictions.

A unique feature of the II_1 factor framework is that it allows a unifying approach to the two types of rigidity phenomena (W^* -rigidity and restrictions on invariants), by considering finite index bimodules between factors (as a generalization of isomorphism between factors and their amplifications). Thus, by explicitly calculating all bimodules between factors in a certain class, one also obtains the fundamental group of the corresponding factors, as well as all possible indices of its subfactors.

In the last section 7, we combine a generalization of Theorem 1.1 with the cocycle superrigidity theorems of [Po05, Po06a] and techniques from [Va07], to give examples of group actions $\Gamma \curvearrowright (X, \mu)$ such that the mere existence of a finite index bimodule between $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$ for an arbitrary free ergodic p.m.p. action $\Lambda \curvearrowright (Y, \eta)$, implies that the groups Γ, Λ are virtually isomorphic and their actions $\Gamma \curvearrowright X, \Lambda \curvearrowright Y$ are virtually conjugate in a very precise sense (see Theorem 7.1 and Example 7.2). In particular, the fundamental group of any of these II_1 factors $L^\infty(X) \rtimes \Gamma$ is trivial and the index of all their subfactors is an integer.

1.3 Comments on the proofs

As we mentioned before, the main difficulty in obtaining W^* -superrigidity lies in proving the uniqueness of the group measure space Cartan decomposition. Indeed, because once such a result is established, W^* -superrigidity can be derived from existing OE-superrigidity results. In our case, 1.2 and 1.3 will follow from our uniqueness of the group measure space Cartan subalgebra in Theorem 1.1 and the OE superrigidity theorems in [Po05, Po06a, Ki09]. By using *intertwining*

subalgebras techniques ([Po01, Po03]), in order to prove the uniqueness, up to unitary conjugacy, of the Cartan subalgebra $A = L^\infty(X)$ of a group measure space factor $M = A \rtimes \Gamma$, it is sufficient to prove that given any other group measure space decomposition $M = B \rtimes \Lambda$, $B = L^\infty(Y)$, there exists a B - A -bimodule $\mathcal{H} \subset L^2(M)$, which is finitely generated over A , a property that we denote by $B \prec_M A$. To get such bimodules, we use the deformation-rigidity theory introduced in [Po01, Po03, Po04] (see [Po06b] for a survey), and in fact the whole array of subsequent developments in [IPP05, PV06, Po06a, Va07, Ho07, CH08], etc. But in order to “locate” the position of the *target* Cartan subalgebra B , with respect to the initial (*source*) Cartan subalgebra A , through these techniques, one needs some amount of rigidity for one of the group actions and a deformation property for the other (like for example in 6.2 of [Po01], 7.1 of [Po04], 7.7 of [IPP05], 1.5 of [Po06a], etc).

Since in W^* -superrigidity statements all assumptions must be on the side of the source Cartan subalgebra $A = L^\infty(X)$, it is thus crucial to show that either the deformation or the rigidity properties of $\Gamma \curvearrowright X$ *automatically transfer* to $\Lambda \curvearrowright Y$. It is precisely the lack of satisfactory “transfer” results that so far prevented from obtaining W^* -superrigidity results. We solve this problem here by proving in Section 2 some very general “transfer of rigidity”, from the source to the target side. While the proofs of these results are quite subtle, let us give here a heuristic explanation.

Assume that $A \rtimes \Gamma = M = B \rtimes \Lambda$ and denote by $(u_g)_{g \in \Gamma}$, resp. $(v_s)_{s \in \Lambda}$, the canonical unitaries in $A \rtimes \Gamma$, resp. $B \rtimes \Lambda$. Every element $x \in M$, has a Fourier expansion $x = \sum_{g \in \Gamma} x_g u_g$ with $x_g \in A$ and we call the x_g the Fourier coefficients of x w.r.t. $\{u_g\}$. We similarly define Fourier coefficients w.r.t. $\{v_s\}$. We assume that $\Gamma_1 < \Gamma$ is a non-amenable subgroup with the relative property (T) and try to transfer this rigidity property to some rigidity for Λ . More precisely, we show that for any deformation ϕ_n of M (i.e. a sequence of c.p. maps on M tending pointwise in the Hilbert norm to id_M), there exist a large n and an infinite subset $\{s_k\}_k \subset \Lambda$, such that $\phi_n(v_{s_k}) \approx v_{s_k}$, $\forall k$, and such that every Fourier coefficient of v_{s_k} w.r.t. $\{u_g\}$ tends to zero in $\|\cdot\|_2$ as $k \rightarrow \infty$. We construct as follows the set $\{s_k\}_k$. The functions $\psi_n(s) = \tau(\phi_n(v_s)v_s^*)$ are positive definite and hence, define the c.p. maps Ψ_n by $\Psi_n(\sum_s b_s v_s) = \sum_s \psi_n(s) b_s v_s$. Since $\Psi_n \rightarrow \text{id}_M$, the relative property (T) of $\Gamma_1 < \Gamma$ ensures that $\Psi_n(u_g) \approx u_g$ uniformly in $g \in \Gamma_1$. This forces $\psi_n(s) \approx 1$ for many of the $s \in \Lambda$ in the support of the Fourier expansion of $u_g, g \in \Gamma_1$ w.r.t. $\{v_s\}$. Among the $s \in \Lambda$ with $\psi_n(s) \approx 1$, we can find a sequence s_k such that the Fourier coefficients of v_{s_k} tend to zero as $k \rightarrow \infty$, because otherwise, it will follow that the $u_g, g \in \Gamma_1$ can roughly be intertwined into A , contradicting the non-amenability of Γ_1 .

As it turns out, if we assume that Γ_1 is freely complemented in Γ , i.e. $\Gamma = \Gamma_1 * \Gamma_2$, then the “tiny” initial information about the group Λ provided by the transfer of rigidity, is enough to prove that $B \prec_M A$. To see this, we first notice that if we apply the above transfer result to the word length deformation $m_\rho(\sum_g a_g u_g) = \sum_g \rho^{|g|} a_g u_g$, as $\rho \rightarrow 1$, then, for ρ close enough to 1, we have $m_\rho(v_{s_k}) \approx v_{s_k}$ uniformly in k . This implies that in the Fourier decomposition with respect to $\{u_g\}$, all v_{s_k} are almost supported by words $g \in \Gamma = \Gamma_1 * \Gamma_2$ of length uniformly bounded by some K . On the other hand, since the Fourier coefficients of v_{s_k} w.r.t. $\{u_g\}$ tend to 0 in $\|\cdot\|_2$, the support of the Fourier expansion of the v_{s_k} lies, as $k \rightarrow \infty$, essentially outside any given finite subset of Γ . If, by contradiction $B \not\prec_M A$, results from [Po03, IPP05] provide a unitary $w \in B$ such that the Fourier expansion of w is essentially supported by a set of words $g \in \Gamma$ of length much larger than K ($|g| \geq 2K$ will do). As we will explain now, this implies that $v_{s_k} w v_{s_k}^* w^*$ and $w^* v_{s_k} w v_{s_k}^*$ are almost orthogonal, contradicting the abelianess of B .

Indeed, first assume for simplicity that all v_{s_k} lie in the span of $Au_g, |g| \leq K$, with the support of

the Fourier expansion of v_{s_k} w.r.t. $\{u_g\}$ tending to infinity in Γ . Similarly, assume that w exactly lies in the span of Au_g , $|g| \geq 2K$. Then, one concludes that all $g \in \Gamma$ in the support of $w^*v_{s_k}wv_{s_k}^*$ eventually have their first K letters in a fixed finite set independent of k , while all $g \in \Gamma$ in the support of $v_{s_k}wv_{s_k}^*w^*$, have their first K letters eventually (as $k \rightarrow \infty$) outside any fixed finite set. In reality, we can only approximate in $\|\cdot\|_2$ and uniformly in k , the unitaries v_{s_k} by elements v'_{s_k} with such good properties. We similarly approximate w by w' . But since our reasoning involves products of 4 elements, the Hilbert norm estimates cannot be handled unless one can control the uniform norms of $w - w'$, $v_{s_k} - v'_{s_k}$. We handle this problem through repeated “trimming” of elements, via Herz-Schur multiplier techniques.

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2 Preliminaries

If $\Gamma \curvearrowright (A, \tau)$ is a trace preserving action of a countable group Γ , we denote by $A \rtimes \Gamma$ the *crossed product* von Neumann algebra, which is the unique tracial von Neumann algebra generated by A and the group of unitaries $(u_g)_{g \in \Gamma}$ satisfying

$$u_g a u_g^* = \sigma_g(a) \quad \text{for all } g \in \Gamma, a \in A \quad \text{and} \quad \tau(a u_g) = \begin{cases} \tau(a) & \text{if } g = e, \\ 0 & \text{if } g \neq e. \end{cases}$$

When $\Gamma \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action, the crossed product $L^\infty(X) \rtimes \Gamma$ is called the *group measure space* II_1 factor associated with $\Gamma \curvearrowright (X, \mu)$. Then, $L^\infty(X)$ is a *Cartan subalgebra* of $L^\infty(X) \rtimes \Gamma$, called a group measure space Cartan subalgebra.

We denote by $L(\Gamma)$ the group von Neumann algebra of a countable group Γ .

Recall that two free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ are called

- *conjugate*, if there exists an isomorphism $\Delta : X \rightarrow Y$ of probability spaces and an isomorphism $\delta : \Gamma \rightarrow \Lambda$ of groups such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ almost everywhere,
- *orbit equivalent*, if there exists an isomorphism $\Delta : X \rightarrow Y$ of probability spaces such that $\Delta(\Gamma \cdot x) = \Lambda \cdot \Delta(x)$ for almost all $x \in X$,
- *W^* -equivalent* (or von Neumann equivalent), if $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$.

If two actions are conjugate, they are obviously orbit equivalent. On the other hand, two actions are orbit equivalent if and only if there exists an isomorphism $L^\infty(X) \rtimes \Gamma \cong L^\infty(Y) \rtimes \Lambda$ sending $L^\infty(X)$ onto $L^\infty(Y)$, see [Si55, FM77].

2.1 Bimodules and weak containment

Let M, N be tracial von Neumann algebras. An M - N -bimodule ${}_M\mathcal{H}_N$ is a Hilbert space \mathcal{H} equipped with a normal representation π of M and a normal anti-representation π' of N such that $\pi(M)$ and $\pi'(N)$ commute.

Given the bimodules ${}_M\mathcal{H}_N$ and ${}_N\mathcal{K}_P$, one can define the *Connes tensor product* $\mathcal{H} \otimes_N \mathcal{K}$ which is an M - P -bimodule, see [Co94, V.Appendix B].

An M - N -bimodule can be seen as well as a representation of the C^* -algebra $M \otimes_{\text{binor}} N^{\text{op}}$. If ${}_M\mathcal{H}_N^1$ and ${}_M\mathcal{H}_N^2$ are M - N -bimodules, we say that \mathcal{H}^1 is *weakly contained* in \mathcal{H}^2 if the corresponding representations π^1, π^2 of $M \otimes_{\text{binor}} N^{\text{op}}$ satisfy $\text{Ker } \pi^1 \supset \text{Ker } \pi^2$. Weak containment behaves well with respect to the Connes tensor product: if ${}_M\mathcal{H}_N^1$ is weakly contained in ${}_M\mathcal{H}_N^2$, then $\mathcal{K} \otimes_M \mathcal{H}^1$ is weakly contained in $\mathcal{K} \otimes_M \mathcal{H}^2$ for every P - M -bimodule \mathcal{K} (see e.g. [An95, Lemma 1.7]). A similar statement holds for tensor products on the right.

We call ${}_ML^2(M)_M$ the *trivial M - M -bimodule* and define the *coarse M - M -bimodule* as the Hilbert space $L^2(M) \otimes L^2(M)$ equipped with the bimodule structure $a \cdot \xi \cdot b = (a \otimes 1)\xi(1 \otimes b)$. A finite von Neumann algebra M is *injective* if the trivial M - M -bimodule is weakly contained in the coarse M - M -bimodule.

The first type of bimodule that we encounter in this article, is the following. Let $\Gamma \curvearrowright (Q, \tau)$ be a trace preserving action and put $M = Q \rtimes \Gamma$. Whenever $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{K})$ is a unitary representation, define the Hilbert space $\mathcal{H}^\pi = L^2(M) \otimes \mathcal{K}$, with bimodule action given by

$$(au_g) \cdot \xi \cdot (bu_h) = (au_g \otimes \pi(g))\xi bu_h \quad \text{for all } a, b \in Q, g, h \in \Gamma.$$

If the unitary representation π is weakly contained in the unitary representation ρ , then \mathcal{H}^π is weakly contained in \mathcal{H}^ρ . If $\pi = \lambda$ is the regular representation of Γ on $\ell^2(\Gamma)$, then $\mathcal{H}^\lambda \cong L^2(M) \otimes_Q L^2(M)$. When Q is injective, ${}_QL^2(Q)_Q$ is weakly contained in ${}_QL^2(Q) \otimes L^2(Q)_Q$ and hence,

$$L^2(M) \otimes_Q L^2(M) \cong L^2(M) \otimes_Q L^2(Q) \otimes_Q L^2(M)$$

is weakly contained in $L^2(M) \otimes_Q (L^2(Q) \otimes L^2(Q)) \otimes_Q L^2(M) = L^2(M) \otimes L^2(M)$. So, for Q injective, \mathcal{H}^λ is weakly contained in the coarse M - M -bimodule. Finally, if Q is injective and if the unitary representation π is weakly contained in the regular representation, then \mathcal{H}^π is weakly contained in the coarse M - M -bimodule.

A next type of bimodule arises from *Jones' basic construction* [Jo83]. Let M be a tracial von Neumann algebra with von Neumann subalgebra P . Denote by $\langle M, e_P \rangle$ the von Neumann algebra acting on $L^2(M)$ generated by M and by the orthogonal projection e_P of $L^2(M)$ onto $L^2(P)$. Equivalently, $\langle M, e_P \rangle$ is the commutant of the right P -action on $L^2(M)$. Given a tracial state τ on M , the von Neumann algebra $\langle M, e_P \rangle$ carries a natural normal semi-finite faithful trace Tr characterized by

$$\text{Tr}(ae_Pb) = \tau(ab) \quad \text{for all } a, b \in M.$$

In particular, we can write the Hilbert space $L^2(\langle M, e_P \rangle)$ and consider it as an M - M -bimodule. Then,

$${}_ML^2(\langle M, e_P \rangle)_M \cong {}_M(L^2(M) \otimes_P L^2(M))_M.$$

Again, if P is injective, it follows that ${}_ML^2(\langle M, e_P \rangle)_M$ is weakly contained in the coarse M - M -bimodule.

2.2 Relative property (T) for an inclusion of finite von Neumann algebras

We recall from [Po01, Proposition 4.1] the following definition of relative property (T) for an inclusion of tracial von Neumann algebras.

Definition 2.1. Let (M, τ) be a tracial von Neumann algebra and let $P \subset M$ be a von Neumann subalgebra. The inclusion $P \subset M$ is said to have the relative property (T) if the following property holds: for every $\varepsilon > 0$, there exists a finite subset $\mathcal{J} \subset M$ and a $\delta > 0$ such that whenever ${}_M\mathcal{H}_M$ is an M - M -bimodule admitting a unit vector ξ with the properties

- $\|a \cdot \xi - \xi \cdot a\| < \delta$ for all $a \in \mathcal{J}$,
- $|\langle \xi, a \cdot \xi \rangle - \tau(a)| < \delta$ and $|\langle \xi, \xi \cdot a \rangle - \tau(a)| < \delta$ for all a in the unit ball of M ,

there exists a vector $\xi_0 \in \mathcal{H}$ satisfying $\|\xi - \xi_0\| < \varepsilon$ and $a \cdot \xi_0 = \xi_0 \cdot a$ for all $a \in P$.

Note that if Γ_0 is a subgroup of the countable group Γ , then the inclusion $L(\Gamma_0) \subset L(\Gamma)$ has relative property (T) if and only if $\Gamma_0 < \Gamma$ has the relative property (T) of Kazhdan-Margulis [Po01, Proposition 5.1].

A normal completely positive map $\varphi : M \rightarrow M$ is said to be subunital if $\varphi(1) \leq 1$ and subtracial if $\tau \circ \varphi \leq \varphi$. Let $P \subset M$ be an inclusion of tracial von Neumann algebras. Relative property (T) then has the following equivalent characterization: whenever $\varphi_n : M \rightarrow M$ is a sequence of subunital and subtracial normal completely positive maps satisfying $\|x - \varphi_n(x)\|_2 \rightarrow 0$ for all $x \in M$, we have that $\|x - \varphi_n(x)\|_2 \rightarrow 0$ uniformly on the unit ball of P .

2.3 Intertwining by bimodules

To fix notations, we briefly recall the *intertwining-by-bimodules* technique from [Po03, Section 2] (see also [Va06, Appendix C]).

Let (M, τ) be a tracial von Neumann algebra and assume that $A, B \subset M_n(\mathbb{C}) \otimes M$ are possibly non-unital von Neumann subalgebras. Denote their respective units by 1_A and 1_B . Then, the following two conditions are equivalent.

- $1_A(M_n(\mathbb{C}) \otimes L^2(M))1_B$ admits an A - B -subbimodule that is finitely generated as a right B -module.
- There is no sequence of unitaries $u_n \in \mathcal{U}(A)$ satisfying $\|E_B(xu_ny^*)\|_2 \rightarrow 0$ for all $x, y \in 1_B(M_n(\mathbb{C}) \otimes M)1_A$.

If one of these equivalent conditions hold, we write $A \prec_M B$. Otherwise, we write $A \not\prec_M B$.

When M is a II_1 factor and $A, B \subset M$ are *Cartan subalgebras*, then $A \prec_M B$ if and only if there exists a unitary $u \in \mathcal{U}(M)$ such that $A = uBu^*$, see [Po01, Theorem A.1] (see also [Va06, Theorem C.3]).

2.4 Cocycle superrigidity

Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action. If \mathcal{L} is a Polish group, an \mathcal{L} -valued 1-cocycle is a measurable map $\omega : \Gamma \times X \rightarrow \mathcal{L}$ satisfying $\omega(gh, x) = \omega(g, h \cdot x)\omega(h, x)$ for all $g, h \in \Gamma$ and almost all $x \in X$. We say that $\Gamma \curvearrowright (X, \mu)$ is \mathcal{L} -cocycle superrigid if every 1-cocycle with values in \mathcal{L} is cohomologous to a group morphism $\Gamma \rightarrow \mathcal{L}$. More precisely, this means that there exists a measurable map $\varphi : X \rightarrow \mathcal{L}$ and a group morphism $\delta : \Gamma \rightarrow \mathcal{L}$ satisfying $\omega(g, x) = \varphi(g \cdot x)\delta(g)\varphi(x)^{-1}$ for all $g \in \Gamma$ and almost all $x \in X$.

We say that $\Gamma \curvearrowright (X, \mu)$ is \mathcal{U}_{fin} -cocycle superrigid if it is \mathcal{L} -cocycle superrigid for all \mathcal{L} in the class \mathcal{U}_{fin} of Polish groups that can be realized as the closed subgroup of the unitary group of a II_1 factor with separable predual. Note that \mathcal{U}_{fin} contains all countable groups and all compact second countable groups. Using the distance given by the $\|\cdot\|_2$ -norm every group in \mathcal{U}_{fin} admits a separable complete bi-invariant metric implementing the topology.

3 Two transfer lemmas

We want to establish W^* -superrigidity for certain known group actions $\Gamma \curvearrowright (X, \mu)$. Starting from a “mysterious” alternative group measure space decomposition $L^\infty(Y) \rtimes \Lambda$ of the given factor $L^\infty(X) \rtimes \Gamma$, we need to transfer certain properties of $\Gamma \curvearrowright X$ to properties of the unknown group Λ or of its unknown action $\Lambda \curvearrowright Y$. For our purposes, this transfer of properties is achieved in Lemmas 3.1 and 3.2.

Lemma 3.1. *Let M be a II_1 factor with the following deformation property. We are given a sequence φ_n of subunital, subtracial, normal, completely positive maps from M to M such that $\|x - \varphi_n(x)\|_2 \rightarrow 0$ for all $x \in M$.*

Let $P \subset M$ be a von Neumann subalgebra and assume that P is injective.

Assume that the countable group Λ acts trace preservingly on the injective von Neumann algebra Q and suppose that we have identified $Q \rtimes \Lambda$ with pMp for some projection $p \in M$. We denote by $(v_s)_{s \in \Lambda}$ the canonical unitaries in $Q \rtimes \Lambda$.

Finally, assume that $M_0 \subset M$ is a von Neumann subalgebra with the relative property (T) and such that M_0 has no injective direct summand.

Then, for every $\varepsilon > 0$, there exists n and a sequence $(s_k)_k$ in Λ such that

1. $\|\varphi_n(v_{s_k}) - v_{s_k}\|_2 \leq \varepsilon$ for all k ,
2. $\|E_P(xv_{s_k}y)\|_2 \rightarrow 0$ for all $x, y \in M$.

Proof. We first argue that we may actually assume that $p = 1$. Put $N = pMp$ and define $\psi_n : N \rightarrow N : \psi_n(a) = p\varphi_n(a)p$. Then, ψ_n is a sequence of subunital, subtracial, normal, completely positive maps and $\|a - \psi_n(a)\|_2 \rightarrow 0$ for all $a \in N$. Let $\varepsilon > 0$. Since $\|p - \varphi_n(p)\|_2 \rightarrow 0$, we can take n_0 such that $\|\psi_n(a) - \varphi_n(a)\|_2 \leq \frac{\varepsilon}{2}\|a\|$ for all $a \in N$ and all $n \geq n_0$. We now replace M by N and φ_n by ψ_n . Since M_0 is diffuse and M is a factor, we may assume that $p \in M_0$ and finally replace M_0 by $pM_0p \subset N$.

So, for the rest of the proof, we assume that $p = 1$. We have $M = Q \rtimes \Lambda$.

Define the positive-definite functions $\tilde{\varphi}_n : \Lambda \rightarrow \mathbb{C} : \tilde{\varphi}_n(s) := \tau(v_s^* \varphi_n(v_s))$. Note that $\tilde{\varphi}_n \rightarrow 1$ pointwise. We have to prove the following statement: for every $\varepsilon > 0$, there exists n and a sequence $(s_k)_k$ in Λ with the properties

$$|\tilde{\varphi}_n(s_k) - 1| \leq \varepsilon \text{ for all } k \quad \text{and} \quad \|E_P(xv_{s_k}y)\|_2 \rightarrow 0 \text{ for all } x, y \in M. \quad (3.1)$$

Suppose that this statement is false. Fix $\varepsilon > 0$ such that for every n , it is impossible to find a sequence $(s_k)_k$ in Λ with properties (3.1).

Define for every n , the normal completely positive map $\theta_n : M \rightarrow M$ satisfying $\theta_n(bv_s) = \tilde{\varphi}_n(s)bv_s$ for all $b \in Q$, $s \in \Lambda$. Note that $\|\theta_n(x) - x\|_2 \rightarrow 0$ for all $x \in M$. Since $M_0 \subset M$ has the relative property (T), fix n such that $\|\theta_n(w) - w\|_2^2 \leq \varepsilon^2/2$ for all $w \in \mathcal{U}(M_0)$.

By assumption, it is impossible to find a sequence $(s_k)_k$ in Λ with properties (3.1). Define

$$\mathcal{V} = \{s \in \Lambda \mid |\tilde{\varphi}_n(s) - 1| \leq \varepsilon\}.$$

It follows that we can take a finite subset $\mathcal{F} \subset M$ and a $\delta > 0$ such that for all $s \in \mathcal{V}$, we have

$$\sum_{x, y \in \mathcal{F}} \|E_P(xv_s y^*)\|_2^2 \geq 2\delta.$$

Consider the Hilbert space $\mathcal{K} = L^2(\langle M, e_P \rangle)$ and the unitary representation

$$\pi : \Lambda \rightarrow \mathcal{U}(\mathcal{K}) : \pi(s)\xi = v_s \xi v_s^* .$$

Since P is injective, the unitary representation π is weakly contained in the regular representation of Λ . Define $\xi_0 \in \mathcal{K}$ by the formula $\xi_0 = \sum_{x \in \mathcal{F}} x^* e_P x$. Define the positive definite function

$$\psi : \Lambda \rightarrow [0, +\infty) : \psi(s) = \langle \xi_0, \pi(s)\xi_0 \rangle .$$

Note that $\psi(s) = \sum_{x,y \in \mathcal{F}} \|E_P(xv_s y)\|_2^2$ and hence, $\psi(s) \geq 2\delta$ for all $s \in \mathcal{V}$.

Define the normal completely positive map $\rho : M \rightarrow M$ such that $\rho(bv_s) = \psi(s)bv_s$ for all $b \in Q$, $s \in \Lambda$. Since Q is injective and since π is weakly contained in the regular representation of Λ , it follows that the M - M -bimodule ${}_M \mathcal{K}^\rho_M$ defined by ρ is weakly contained in the coarse M - M -bimodule ${}_M(L^2(M) \otimes L^2(M))_M$.

We claim that $\tau(w^* \rho(w)) \in [\delta, 1]$ for all $w \in \mathcal{U}(M_0)$. Since $\|\theta_n(w) - w\|_2^2 \leq \varepsilon^2/2$ for all $w \in \mathcal{U}(M_0)$, it suffices to prove that $\tau(w^* \rho(w)) \in [\delta, 1]$ for every unitary $w \in \mathcal{U}(M)$ satisfying $\|\theta_n(w) - w\|_2^2 \leq \varepsilon^2/2$. Take such a unitary w and write $w = \sum_{s \in \Lambda} b_s v_s$. It follows that

$$\begin{aligned} \frac{\varepsilon^2}{2} &\geq \|\theta_n(w) - w\|_2^2 = \sum_{s \in \Lambda} |\tilde{\varphi}_n(s) - 1|^2 \|b_s\|_2^2 \\ &\geq \sum_{s \in \Lambda - \mathcal{V}} |\tilde{\varphi}_n(s) - 1|^2 \|b_s\|_2^2 \\ &\geq \varepsilon^2 \sum_{s \in \Lambda - \mathcal{V}} \|b_s\|_2^2 . \end{aligned}$$

We conclude that

$$\sum_{s \in \Lambda - \mathcal{V}} \|b_s\|_2^2 \leq \frac{1}{2} .$$

Since w is unitary, it follows that

$$\sum_{s \in \mathcal{V}} \|b_s\|_2^2 \geq \frac{1}{2} .$$

It now follows that

$$\begin{aligned} \tau(w^* \rho(w)) &= \sum_{s \in \Lambda} \psi(s) \|b_s\|_2^2 \geq \sum_{s \in \mathcal{V}} \psi(s) \|b_s\|_2^2 \\ &\geq 2\delta \sum_{s \in \mathcal{V}} \|b_s\|_2^2 \geq \delta . \end{aligned}$$

Hence, our claim is proven.

Since $\tau(w^* \rho(w)) \in [\delta, 1]$ for all $w \in \mathcal{U}(M_0)$, the M - M -bimodule ${}_M \mathcal{K}^\rho_M$ contains a non-zero M_0 -central vector. On the other hand, the M - M -bimodule ${}_M \mathcal{K}^\rho_M$ is weakly contained in the coarse M - M -bimodule ${}_M(L^2(M) \otimes L^2(M))_M$. It follows that the coarse M_0 - M_0 -bimodule admits a sequence of almost M_0 -central unit vectors. This is a contradiction with the assumption that M_0 has no injective direct summand. \square

Lemma 3.2. *Let M be a II_1 factor with the following deformation property. We have an inclusion $M \subset \tilde{M}$ of M into the finite von Neumann algebra \tilde{M} such that the M - M -bimodule*

${}_M(\widetilde{L^2(\widetilde{M})} \oplus L^2(M))_M$ is weakly contained in the coarse M - M -bimodule ${}_M(L^2(M) \otimes L^2(M))_M$. We are given a sequence $\alpha_n \in \text{Aut}(\widetilde{M})$ such that $\|a - \alpha_n(a)\|_2 \rightarrow 0$ for all $a \in \widetilde{M}$.

Let $P \subset M$ be a von Neumann subalgebra and assume that P is injective.

Assume that the countable group Λ acts trace preservingly on the injective von Neumann algebra Q and suppose that we have identified $Q \rtimes \Lambda = pMp$ for some projection $p \in M$. We denote by $(v_s)_{s \in \Lambda}$ the canonical unitaries in $Q \rtimes \Lambda$.

Finally, assume that $M_1, M_2 \subset M$ are von Neumann subalgebras such that M_1 and M_2 commute and M_1, M_2 have no injective direct summand.

Then, for every $\varepsilon > 0$, there exists n and a sequence $s_k \in \Lambda$ such that

1. $\|\alpha_n(v_{s_k}) - E_M(\alpha_n(v_{s_k}))\|_2 \leq \varepsilon$ for all k ,
2. $\|E_P(xv_{s_k}y)\|_2 \rightarrow 0$ for all $x, y \in M$.

Proof. We use throughout the trace τ normalized in such a way that $\tau(p) = 1$. We use the notation $\|\cdot\|_2$ accordingly.

Assume that the statement is false. Take $\varepsilon > 0$ such that for all $n \in \mathbb{N}$, it is impossible to find a sequence $s_k \in \Lambda$ satisfying properties 1 and 2 above.

Put $N := Q \rtimes \Lambda$, which we identified with pMp . Define the normal, unital $*$ -homomorphism $\beta : N \rightarrow N \otimes N$ given by $\beta(av_s) = av_s \otimes v_s$ for all $a \in Q$, $s \in \Lambda$. Put $\widetilde{N} := p\widetilde{M}p$. Define the N - N -bimodule ${}_N\mathcal{H}_N$ given by $\mathcal{H} := L^2(N) \otimes L^2(\widetilde{N})$ with $x \cdot \xi \cdot y := \beta(x)\xi\beta(y)$. Put $\mathcal{H}^0 := L^2(N) \otimes (L^2(\widetilde{N}) \oplus L^2(N))$. Note that $\mathcal{H}^0 \subset \mathcal{H}$ is an N - N -subbimodule. By our assumptions and because Q is injective, the N - N -bimodule ${}_N\mathcal{H}_N^0$ is weakly contained in the coarse N - N -bimodule.

Since M_1 is diffuse and M is a factor, we may assume that $p \in M_1$. We replace M_1 by pM_1p and M_2 by M_2p so that they become commuting subalgebras of N .

Put $\varepsilon_1 = \frac{1}{10}\varepsilon$. Since M_1 has no injective direct summand and since ${}_{M_1}\mathcal{H}_{M_1}^0$ is weakly contained in the coarse M_1 - M_1 -bimodule, we can take a finite subset $\mathcal{F} \subset M_1$ and a $\rho > 0$ such that whenever $\xi \in \mathcal{H}^0$ and $\|x \cdot \xi - \xi \cdot x\|_2 \leq \rho$ for all $x \in \mathcal{F}$, then $\|\xi\|_2 \leq \varepsilon_1$.

Define the linear map

$$\theta_n : M_2 \rightarrow \mathcal{H} : \theta_n(a) = (1 \otimes p)(\text{id} \otimes \alpha_n)\beta(a)(1 \otimes p).$$

Take n large enough such that

$$\|\alpha_n(p) - p\|_2 \quad \text{and} \quad \|(\text{id} \otimes \alpha_n^{-1})\beta(x) - \beta(x)\|_2, x \in \mathcal{F},$$

are all sufficiently small in order to ensure that

$$\|x \cdot \theta_n(a) - \theta_n(a) \cdot x\|_2 \leq \rho \quad \text{for all } x \in \mathcal{F}, a \in \mathcal{U}(M_2).$$

It follows that

$$\|\theta_n(a) - (\text{id} \otimes E_M)\theta_n(a)\|_2 \leq \varepsilon_1 \quad \text{for all } a \in \mathcal{U}(M_2).$$

When choosing n , we can make sure that $\|\alpha_n(p) - p\|_2 \leq \varepsilon_1$, yielding

$$\|\theta_n(a) - (\text{id} \otimes \alpha_n)\beta(a)\|_2 \leq 2\varepsilon_1\|a\| \quad \text{for all } a \in M_2.$$

As a conclusion, we get

$$\|(\text{id} \otimes \alpha_n)\beta(a) - (\text{id} \otimes E_M \circ \alpha_n)\beta(a)\|_2 \leq 5\varepsilon_1 \quad \text{for all } a \in \mathcal{U}(M_2). \quad (3.2)$$

Define $\mathcal{V} = \{s \in \Lambda \mid \|\alpha_n(v_s) - E_M(\alpha_n(v_s))\|_2 \leq \varepsilon\}$. By our assumption ex absurdo, it is impossible to find a sequence $s_k \in \Lambda$ satisfying conditions 1 and 2 in the formulation of the lemma. Hence, we find a finite subset $\mathcal{F}_1 \subset M$ and a $\delta > 0$ such that for all $s \in \mathcal{V}$, we have

$$\sum_{x,y \in \mathcal{F}_1} \|E_P(xv_sy^*)\|_2^2 \geq \delta.$$

Define the N - N -bimodule ${}_N\mathcal{K}_N$ where

$$\mathcal{K} = L^2(N) \otimes pL^2(\langle M, e_P \rangle)p \quad \text{and} \quad x \cdot \xi \cdot y = \beta(x)\xi\beta(y).$$

Since P is injective, ${}_N(pL^2(\langle M, e_P \rangle)p)_N$ is contained in the coarse N - N -bimodule. Since Q is injective, also ${}_N\mathcal{K}_N$ is weakly contained in the coarse N - N -bimodule. We now prove that \mathcal{K} admits a non-zero M_2 -central vector. Since M_2 has no injective direct summand, this yields the required contradiction.

Define the vector $\xi \in \mathcal{K}$ by the formula

$$\xi := \sum_{x \in \mathcal{F}_1} 1 \otimes px^*e_Pxp.$$

We will prove that

$$\langle a \cdot \xi \cdot a^*, \xi \rangle \geq \frac{3}{4}\delta \quad \text{for all } a \in \mathcal{U}(M_2). \quad (3.3)$$

Take $a \in \mathcal{U}(M_2)$ and write $a = \sum_{s \in \Lambda} a_s v_s$ with $a_s \in Q$. Then,

$$(\text{id} \otimes \alpha_n)\beta(a) - (\text{id} \otimes E_M \circ \alpha_n)\beta(a) = \sum_{s \in \Lambda} a_s v_s \otimes (\alpha_n(v_s) - E_M(\alpha_n(v_s))).$$

Using (3.2) and the definition of $\mathcal{V} \subset \Lambda$, it follows that

$$\begin{aligned} (5\varepsilon_1)^2 &\geq \|(\text{id} \otimes \alpha_n)\beta(a) - (\text{id} \otimes E_M \circ \alpha_n)\beta(a)\|_2^2 \\ &= \sum_{s \in \Lambda} \|a_s\|_2^2 \|\alpha_n(v_s) - E_M(\alpha_n(v_s))\|_2^2 \\ &\geq \varepsilon^2 \sum_{s \in \Lambda \setminus \mathcal{V}} \|a_s\|_2^2. \end{aligned}$$

Using our definition of ε_1 , we conclude that

$$\sum_{s \in \Lambda \setminus \mathcal{V}} \|a_s\|_2^2 \leq \frac{1}{4} \quad \text{and hence} \quad \sum_{s \in \mathcal{V}} \|a_s\|_2^2 \geq \frac{3}{4}.$$

It follows that

$$\begin{aligned} \langle a \cdot \xi \cdot a^*, \xi \rangle &= \sum_{s \in \Lambda} \|a_s\|_2^2 \left(\sum_{x,y \in \mathcal{F}_1} \|E_P(xv_sy^*)\|_2^2 \right) \\ &\geq \delta \sum_{s \in \mathcal{V}} \|a_s\|_2^2 \geq \frac{3}{4}\delta. \end{aligned}$$

So, we have shown (3.3). It follows that the unique vector of minimal norm in the closed convex hull of $\{a \cdot \xi_0 \cdot a^* \mid a \in \mathcal{U}(M_2)\}$ is non-zero and M_2 -central. This ends the proof of the lemma. \square

4 Some Herz-Schur multipliers on amalgamated free products

Let Γ be a countable group. A function $\varphi : \Gamma \rightarrow \mathbb{C}$ is called a *Herz-Schur multiplier* if $u_g \mapsto \varphi(g)u_g$ extends to an ultraweakly continuous, completely bounded linear map $m_\varphi : L(\Gamma) \rightarrow L(\Gamma)$. The linear space of Herz-Schur multipliers on Γ is denoted by $B_2(\Gamma)$. Whenever $\varphi \in B_2(\Gamma)$, put $\|\varphi\|_{cb} := \|m_\varphi\|_{cb}$.

Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be an amalgamated free product. All elements $g \in \Gamma$ have a natural *length* $|g|$, arising by writing g as an alternating product of elements $\Gamma_1 - \Sigma$ and elements in $\Gamma_2 - \Sigma$. By convention, $|g| = 0$ if and only if $g \in \Sigma$.

The following result is probably well known. It is an immediate consequence of [BP93, Proposition 3.2] and we include a proof for the convenience of the reader.

Lemma 4.1. *1. For every $K \in \mathbb{N}$, there exists $K' > K$ and $\psi \in B_2(\Gamma)$ with the following properties.*

- $\|\psi\|_{cb} \leq 2$.
- $\psi(g) = 1$ if $|g| \leq K$ and $\psi(g) = 0$ if $|g| \geq K'$.
- $0 \leq \psi(g) \leq 1$ for all $g \in \Gamma$.

2. For every $K \in \mathbb{N}$, there exists $K' > K$ and $\varphi \in B_2(\Gamma)$ with the following properties.

- $\|\varphi\|_{cb} \leq 3$.
- $\varphi(g) = 0$ if $|g| \leq K$ and $\varphi(g) = 1$ if $|g| \geq K'$.
- $0 \leq \varphi(g) \leq 1$ for all $g \in \Gamma$.

Proof. 1. Whenever $0 < \rho < 1$, define $\theta_\rho(g) = \rho^{|g|}$. For all $n \in \mathbb{N}$, put $\gamma_n(g) = 1$ if $|g| = n$ and $\gamma_n(g) = 0$ if $|g| \neq n$. By [BP93, Proposition 3.2], all θ_ρ and γ_n belong to $B_2(\Gamma)$ and satisfy $\|\theta_\rho\|_{cb} = 1$ and $\|\gamma_n\|_{cb} \leq 4n + 1$.

Choose $K \in \mathbb{N}$. Take $0 < \rho < 1$ close enough to 1 such that

$$\sum_{n=0}^K (1 - \rho^n)(4n + 1) \leq \frac{1}{2}.$$

Next, take $K' > K$ large enough such that

$$\sum_{n=K'}^{\infty} \rho^n(4n + 1) \leq \frac{1}{2}.$$

Define

$$\psi(g) = \begin{cases} 1 & \text{if } |g| \leq K, \\ \rho^{|g|} & \text{if } K < |g| < K', \\ 0 & \text{if } |g| \geq K'. \end{cases}$$

Since

$$\psi = \theta_\rho + \sum_{n=0}^K (1 - \rho^n)\gamma_n - \sum_{n=K'}^{\infty} \rho^n \gamma_n,$$

we conclude that $\psi \in B_2(\Gamma)$ and $\|\psi\|_{cb} \leq 2$.

2. Take ψ as in 1 and put $\varphi(g) = 1 - \psi(g)$. □

5 Factors with unique group measure space Cartan subalgebra

Definition 5.1. We define the family \mathcal{G} of groups Γ of the form $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ with the following properties.

1. Γ_1 contains a non-amenable subgroup with the relative property (T) or Γ_1 contains two non-amenable commuting subgroups,
2. Σ is amenable and $\Gamma_2 \neq \Sigma$,
3. There exist $g_1, \dots, g_k \in \Gamma$ such that $\bigcap_{i=1}^k g_i \Sigma g_i^{-1}$ is finite.

Our main result says that all group measure space II_1 factors with groups $\Gamma \in \mathcal{G}$ have a unique group measure space Cartan subalgebra. We also deal with amplifications. Therefore, denote by $D_n(\mathbb{C})$ the subalgebra of diagonal matrices in $M_n(\mathbb{C})$. The following is a more general version of Theorem 1.1.

Theorem 5.2. *Let Γ be a group in the family \mathcal{G} and $\Gamma \curvearrowright (X, \mu)$ a free ergodic p.m.p. action. Denote $M = L^\infty(X) \rtimes \Gamma$. Whenever $\Lambda \curvearrowright (Y, \eta)$ is a free ergodic p.m.p. action, $p \in M_n(\mathbb{C}) \otimes M$ is a projection and*

$$\pi : L^\infty(Y) \rtimes \Lambda \rightarrow p(M_n(\mathbb{C}) \otimes M)p$$

is an isomorphism, there exists a projection $q \in D_n(\mathbb{C}) \otimes L^\infty(X)$ and a unitary $u \in q(M_n(\mathbb{C}) \otimes M)p$ such that

$$\pi(L^\infty(Y)) = u^*(D_n(\mathbb{C}) \otimes L^\infty(X))u.$$

In the rest of this section, we prove Theorems 5.2 and 1.4. We already deduce the following result, similar to [PV08c, Theorem 1.2] which dealt with certain free product groups $\Gamma = \Gamma_1 * \Gamma_2$.

Corollary 5.3. *Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be a group in the family \mathcal{G} . Assume that Γ_1 and Γ_2 are finitely generated and assume that at least one of the Γ_i has fixed prize with cost strictly larger than 1 (e.g. $\Gamma_2 = \mathbb{F}_n$ for $2 \leq n < \infty$).*

For any free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$, the II_1 factor $L^\infty(X) \rtimes \Gamma$ has trivial fundamental group. In other words, using the notation of [PV08c], we have $\mathcal{S}_{\text{factor}}(\Gamma) = \{\{1\}\}$.

Proof. Theorem 5.2 implies that the fundamental group of $L^\infty(X) \rtimes \Gamma$ equals the fundamental group of the orbit equivalence relation $\mathcal{R} = \mathcal{R}(\Gamma \curvearrowright X)$. By [Ga00, Théorème IV.15], \mathcal{R} has cost strictly between 1 and ∞ . It then follows from [Ga00, Proposition II.6] that \mathcal{R} has trivial fundamental group. \square

5.1 Deformation of amalgamated free product factors

Let M_1, M_2 be von Neumann algebras equipped with a faithful normal tracial state τ . Assume that P is a common von Neumann subalgebra of M_1 and M_2 and that the traces of M_1, M_2 coincide on P . Denote by $M = M_1 *_P M_2$ the amalgamated free product with respect to the unique trace preserving conditional expectations (see [Po93] and [VDN92]) and still denote by τ the canonical

tracial state on M . The Hilbert space $L^2(M)$ can be explicitly realized as follows, where we denote $L^2(M_i^\circ) := L^2(M_i) \ominus L^2(P)$.

$$L^2(M) = L^2(P) \oplus \bigoplus_{i_1 \neq i_2, i_2 \neq i_3, \dots, i_{n-1} \neq i_n} (L^2(M_{i_1}^\circ) \otimes_P L^2(M_{i_2}^\circ) \otimes_P \cdots \otimes_P L^2(M_{i_n}^\circ)) .$$

For all $0 < \rho < 1$, denote by $m_\rho \in B(L^2(M))$ the operator given by multiplication with the positive scalar ρ^n on $L^2(M_{i_1}^\circ) \otimes_P L^2(M_{i_2}^\circ) \otimes_P \cdots \otimes_P L^2(M_{i_n}^\circ)$. Actually (and this will incidentally be a consequence from the following discussion), there is a unique normal unital completely positive map $m_\rho : M \rightarrow M$ whose extension to $L^2(M)$ is the m_ρ that we have just defined.

Following [IPP05, Section 2.2], we can define the following deformation of M . Define, for $i = 1, 2$, $\widetilde{M}_i := M_i *_P (P \overline{\otimes} L(\mathbb{Z}))$. Define $\widetilde{M} := \widetilde{M}_1 *_P \widetilde{M}_2$ and observe that we have a canonical identification $\widetilde{M} = M *_P (P \overline{\otimes} L(\mathbb{F}_2))$. We now define a one-parameter group of automorphisms α_t of \widetilde{M} .

Whenever $\alpha \in \text{Aut}(\widetilde{M}_1)$ and $\beta \in \text{Aut}(\widetilde{M}_2)$ are both the identity when restricted to P , we have a unique $\alpha * \beta \in \text{Aut}(\widetilde{M})$ simultaneously extending α and β . Denote by u the canonical unitary generator of $L(\mathbb{Z})$ viewed as a subalgebra of \widetilde{M}_1 and denote by v the canonical unitary generator of $L(\mathbb{Z})$ viewed as a subalgebra of \widetilde{M}_2 . Let $f : \mathbb{T} \rightarrow (-\pi, \pi]$ be the unique map satisfying $z = \exp(if(z))$ for all $z \in \mathbb{T}$. Define the self-adjoint elements $h \in \widetilde{M}_1$ and $k \in \widetilde{M}_2$ as $h = f(u)$ and $k = f(v)$. Put, for all $t \in \mathbb{R}$, $u_t = \exp(ith)$ and $v_t = \exp(itk)$. Then, $\text{Ad}(u_t) \in \text{Aut}(\widetilde{M}_1)$ and $\text{Ad}(v_t) \in \text{Aut}(\widetilde{M}_2)$ are both the identity on P , so that we can define $\alpha_t \in \text{Aut}(\widetilde{M})$ as $\alpha_t := \text{Ad}(u_t) * \text{Ad}(v_t)$.

Define $\rho_t = \tau(u_t)^2 = \tau(v_t)^2$. It happens to be that $\rho_t = \frac{\sin^2(\pi t)}{(\pi t)^2}$ and hence, if t decreases from 1 to 0, then ρ_t increases from 0 to 1. By using the definition of the trace on M , it is easy to see that

$$E_M(\alpha_t(x)) = m_{\rho_t}(x) .$$

This also shows that m_ρ is a normal unital completely positive map on M for all $0 < \rho < 1$.

Observe that, when P is injective, the M - M -bimodule ${}_M(L^2(\widetilde{M}) \ominus L^2(M))_M$ is weakly contained in the coarse M - M -bimodule ${}_M(L^2(M) \otimes L^2(M))_M$ (see [CH08, Proposition 3.1] for a detailed argument). So, we are then in a situation where Lemma 3.2 can potentially be applied.

We now present one of the main technical results from [IPP05]. Since the statement as we need it, is not exactly formulated in [IPP05], we give a sketch of proof, indicating the different steps from [IPP05] that are needed to prove the result. Recall that the *normalizer* $\mathcal{N}_N(Q)''$ of a von Neumann subalgebra $Q \subset N$ is the von Neumann algebra generated by the group of unitaries $u \in N$ satisfying $uQu^* = Q$.

Theorem 5.4. *Let M_1, M_2 be tracial von Neumann algebras with a common von Neumann subalgebra P on which the traces coincide. Denote by $M = M_1 *_P M_2$ the amalgamated free product w.r.t. the trace preserving conditional expectations. Denote, for $0 < \rho < 1$, by m_ρ the completely positive map on M introduced above. Let $p \in M$ be a projection and $Q \subset pMp$ a von Neumann subalgebra.*

If there exists $0 < \rho < 1$ and $\delta > 0$ such that $\tau(v^ m_\rho(v)) \geq \delta$ for all $v \in \mathcal{U}(Q)$, then $Q \prec_M P$ or $\mathcal{N}_{pMp}(Q)'' \prec_M M_i$ for some $i \in \{1, 2\}$. As explained above, $\mathcal{N}_{pMp}(Q)''$ denotes the normalizer of Q inside pMp .*

Proof. Clearly, $\tau(v^* m_\rho(v))$ increases when ρ increases. So, with the notation introduced before the theorem, we can take $0 < t_0 < 1$ such that

$$\tau(v^* \alpha_t(v)) = \tau(v^* E_M(\alpha_t(v))) = \tau(v^* m_{\rho_t}(v)) \geq \delta \quad \text{for all } v \in \mathcal{U}(Q), \quad 0 < t \leq t_0 .$$

In particular, we can take t of the form $t = 2^{-n}$ such that $\tau(v^* \alpha_t(v)) \geq \delta$ for all $v \in \mathcal{U}(Q)$. Denote by $y \in p\widetilde{M}\alpha_t(p)$ the unique element of minimal $\|\cdot\|_2$ in the closed convex hull of $\{v^* \alpha_t(v) \mid v \in \mathcal{U}(Q)\}$. It follows that $y \neq 0$ because $\tau(y) \geq \delta$ and that $xy = y\alpha_t(x)$ for all $x \in Q$.

Assume that $Q \not\prec_M P$. We have to prove that $\mathcal{N}_{pMp}(Q)'' \prec_M M_i$ for some $i \in \{1, 2\}$. Repeating [IPP05, Proof of 3.3] (see also [Ho07, Step (2) of 5.6]), we find a non-zero $z \in p\widetilde{M}\alpha_1(p)$ satisfying $xz = z\alpha_1(x)$ for all $x \in Q$. As in [IPP05, Proof of 4.3] (actually, literally repeating [Ho07, Step (3) of 5.6]), we can conclude that for $i = 1$ or $i = 2$, we have $Q \prec_M M_i$. Since we assumed that $Q \not\prec_M P$, [IPP05, Theorem 1.1] (see also [Ho07, Theorem 4.6]) implies that the normalizer $\mathcal{N}_{pMp}(Q)''$ can be embedded into M_i inside M . \square

For later use, we also record the following non-optimal inequality. As positive operators on $L^2(M)$, we have $(1 - m_\rho)^2 \leq 1 - m_\rho \leq 1 - m_\rho^2$. Hence, $\|x - m_\rho(x)\|_2^2 \leq \|x\|_2^2 - \|m_\rho(x)\|_2^2$. On the other hand, for all $t \in \mathbb{R}$ and $x \in M$, we have

$$\|\alpha_t(x) - E_M(\alpha_t(x))\|_2^2 = \|x\|_2^2 - \|E_M(\alpha_t(x))\|_2^2 = \|x\|_2^2 - \|m_{\rho_t}(x)\|_2^2.$$

It follows that

$$\|x - m_{\rho_t}(x)\|_2 \leq \|\alpha_t(x) - E_M(\alpha_t(x))\|_2 \quad \text{for all } x \in M, 0 < t < 1. \quad (5.1)$$

Finally, let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be an amalgamated free product of groups. Assume that Γ acts in a trace preserving way on the tracial von Neumann algebra (A, τ) . Put $M = A \rtimes \Gamma$. Putting $M_i = A \rtimes \Gamma_i$ for $i = 1, 2$ and $P = A \rtimes \Sigma$, we have $M = M_1 *_P M_2$ in a canonical way. Moreover, for all $0 < \rho < 1$, $a \in A$, $g \in \Gamma$, we have

$$m_\rho(au_g) = \rho^{|g|} au_g,$$

where $|g|$ denotes the length of g in the sense explained at the beginning of Section 4.

5.2 A combinatorial lemma

Fix an amalgamated free product $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ and a trace preserving action $\Gamma \curvearrowright (A, \tau)$ of Γ on the tracial von Neumann algebra (A, τ) . Put $M = A \rtimes \Gamma$ and $P = A \rtimes \Sigma$.

For every $K \in \mathbb{N}$, denote by \mathcal{P}_K the orthogonal projection of $L^2(M)$ onto the closed linear span of $\{au_g \mid |g| \leq K, a \in A\}$.

Whenever $g_0, h_0 \in (\Gamma_1 - \Sigma) \cup (\Gamma_2 - \Sigma)$, denote by W_{g_0, h_0} the subset of Γ consisting of those $g \in \Gamma$ with $|g| \geq 2$ admitting a reduced expression starting with g_0 and ending with h_0 (this means that any reduced expression for g starts with $g_0\sigma$ and ends with $\sigma'h_0$ for some $\sigma, \sigma' \in \Sigma$). Denote by \mathcal{P}_{g_0, h_0} the orthogonal projection of $L^2(M)$ onto the closed linear span of $\{au_g \mid g \in W_{g_0, h_0}, a \in A\}$.

In general, whenever $W \subset \Gamma$, denote by \mathcal{P}_W the orthogonal projection of $L^2(M)$ onto the closed linear span of $\{au_g \mid g \in W, a \in A\}$.

Lemma 5.5. *Let $K \in \mathbb{N}$ and assume that (y_k) is a bounded sequence in M with the following properties.*

- $y_k = \mathcal{P}_K(y_k)$ for all k .
- $\|E_P(xy_kz)\|_2 \rightarrow 0$ for all $x, z \in M$.

Let $g, h \in \Gamma$ with $|g|, |h| \geq K$ and write g, h as reduced expressions. Denote by g_0 the first letter of g and by h_0 the last letter of h . Then, we can write

$$u_g y_k u_h = a_k + b_k$$

where a_k, b_k are bounded sequences in M satisfying the following properties.

- $a_k = \mathcal{P}_{g_0, h_0}(a_k)$ for all k .
- $\|b_k\|_2 \rightarrow 0$.

Proof. We fix once and for all words with letters alternatingly from $\Gamma_1 - \Sigma$ and $\Gamma_2 - \Sigma$ representing the elements g and h .

Let $(g_1, h_1), \dots, (g_N, h_N)$ be an enumeration of all pairs of words (g', h') satisfying

- $|g'| + |h'| \leq K$,
- the fixed word representing g ends with the subword g' ,
- the fixed word representing h starts with the subword h' .

Define $W_i = g g_i^{-1} \Sigma h_i^{-1} h$ and $W = \bigcup_{i=1}^N W_i$. Observe that

$$\mathcal{P}_{W_i}(x) = u_g u_{g_i}^* E_P(u_{g_i} u_g^* x u_h^* u_{h_i}) u_{h_i}^* u_h$$

for all $x \in M$. Hence, \mathcal{P}_{W_i} is completely bounded as a map from M to M . The orthogonal projections \mathcal{P}_{W_i} commute and hence

$$1 - \mathcal{P}_W = (1 - \mathcal{P}_{W_1}) \cdots (1 - \mathcal{P}_{W_N}).$$

So, \mathcal{P}_W is completely bounded on M . We put $b_k = \mathcal{P}_W(u_g y_k u_h)$ and $a_k = u_g y_k u_h - b_k$. So, a_k and b_k are bounded sequences in M with $u_g y_k u_h = a_k + b_k$.

First observe that

$$\|b_k\|_2^2 \leq \sum_{i=1}^N \|\mathcal{P}_{W_i}(u_g y_k u_h)\|_2^2 = \sum_{i=1}^N \|E_P(u_{g_i} y_k u_{h_i})\|_2^2 \rightarrow 0.$$

It remains to prove that $a_k = \mathcal{P}_{g_0, h_0}(a_k)$ for all k . But, if $r \in \Gamma$ with $|r| \leq K$ and if grh admits no reduced expression that starts with g_0 and ends with h_0 , there must exist $i \in \{1, \dots, N\}$ such that $g_i r h_i \in \Sigma$ and hence $grh \in W$. As a consequence, whenever $y \in M$ with $y = \mathcal{P}_K(y)$, we have

$$u_g y u_h - \mathcal{P}_W(u_g y u_h) \in \mathcal{P}_{g_0, h_0}(L^2(M)).$$

This concludes the proof of the lemma. □

5.3 Group measure space Cartan subalgebras can be intertwined into $A \rtimes \Sigma$

Fix $\Gamma = \Gamma_1 *_{\Sigma} \Gamma_2$ satisfying the conditions 1 and 2 of Definition 5.1. Let $\Gamma \curvearrowright (A, \tau)$ be a trace preserving action of Γ on the tracial injective von Neumann algebra (A, τ) . Put $M = A \rtimes \Gamma$ and $P = A \rtimes \Sigma$. Note that P is injective, because A is injective and Σ is amenable.

Theorem 5.6. *Whenever $p \in M$ is a non-zero projection and $pMp = B \rtimes \Lambda$ is a crossed product decomposition where $\Lambda \curvearrowright (B, \tau)$ is a trace preserving action on the abelian von Neumann algebra B , then $B \prec_M P$.*

We will prove Theorem 5.6 by combining the transfer of rigidity lemmas 3.1, 3.2 with the following result saying that any abelian algebra that is normalized by ‘many’ unitaries of short word length, is itself uniformly of short length.

As in Section 5.1, define for every $0 < \rho < 1$, the unital completely positive map m_ρ on M by $m_\rho(au_g) = \rho^{|g|}au_g$ for all $a \in A$, $g \in \Gamma$.

Lemma 5.7. *Let $p \in M$ be a projection and $B \subset pMp$ an abelian von Neumann subalgebra. Put $\varepsilon = \tau(p)/2072$ and assume that we are given $0 < \rho < 1$ and a sequence of unitaries $v_k \in pMp$ that normalize B and satisfy*

- $\|v_k - m_\rho(v_k)\|_2 \leq \varepsilon/2$ for all k ,
- $\|E_P(xv_ky)\|_2 \rightarrow 0$ for all $x, y \in M$.

*Then, there exists a $0 < \rho_0 < 1$ and a $\delta > 0$ such that $\tau(w^*m_{\rho_0}(w)) \geq \delta$ for all $w \in \mathcal{U}(B)$.*

Proof. Throughout the proof we make use of the Herz-Schur multipliers provided by Lemma 4.1. Whenever $\varphi \in B_2(\Gamma)$, we can extend m_φ to $A \rtimes \Gamma$, without increasing the cb-norm, by putting $m_\varphi(au_g) = \varphi(g)au_g$ for all $a \in A$ and $g \in \Gamma$.

Assume that the lemma is false. Take $K \in \mathbb{N}$ such that $\rho^K \leq 1/2$. Denote as before by \mathcal{P}_K the orthogonal projection of $L^2(M)$ onto the closed linear span of $\{au_g \mid a \in A, |g| \leq K\}$. Note that

$$\|v - \mathcal{P}_K(v)\|_2 \leq 2\|m_\rho(v) - v\|_2$$

for all $v \in M$.

By Lemma 4.1.1, take $K_1 > K$ and $\psi \in B_2(\Gamma)$ satisfying $\|\psi\|_{cb} \leq 2$, $\psi(g) = 1$ if $|g| \leq K$, $\psi(g) = 0$ if $|g| \geq K_1$ and $0 \leq \psi(g) \leq 1$ for all $g \in \Gamma$. Define $y_k := m_\psi(v_k)$. Since

$$\|v_k - \mathcal{P}_K(v_k)\|_2 \leq 2\|m_\rho(v_k) - v_k\|_2 \leq \varepsilon,$$

we get $\|v_k - y_k\|_2 \leq \varepsilon$. So, the sequence (y_k) satisfies

- $\|y_k\| \leq 2$ for all k ,
- $y_k = \mathcal{P}_{K_1}(y_k)$ for all k ,
- $\|v_k - y_k\|_2 \leq \varepsilon$.

We claim that also $\|E_P(xy_kz)\|_2 \rightarrow 0$ for all $x, z \in M$. Since (y_k) is a bounded sequence and since $A \subset P$, we may assume that $x = u_g, z = u_h$ for some $g, h \in \Gamma$. Denote by \mathcal{P}_0 the orthogonal projection of $L^2(M)$ onto the closed linear span of $\{au_r \mid a \in A, r \in g^{-1}\Sigma h^{-1}\}$. Then,

$$\begin{aligned} \|E_P(u_g y_k u_h)\|_2 &= \|\mathcal{P}_0(y_k)\|_2 = \|\mathcal{P}_0(\mathfrak{m}_\psi(v_k))\|_2 \\ &= \|\mathfrak{m}_\psi(\mathcal{P}_0(v_k))\|_2 \leq \|\mathcal{P}_0(v_k)\|_2 \\ &= \|E_P(u_g v_k u_h)\|_2 \rightarrow 0. \end{aligned}$$

This proves the claim.

By Lemma 4.1.2, take $K_2 > 2K_1$ and $\varphi \in B_2(\Gamma)$ satisfying $\|\varphi\|_{\text{cb}} \leq 3$, $\varphi(g) = 0$ if $|g| \leq 2K_1$, $\varphi(g) = 1$ if $|g| \geq K_2$ and $0 \leq \varphi(g) \leq 1$ for all $g \in \Gamma$.

Take $0 < \rho_0 < 1$ close enough to 1 such that $\rho_0^{K_2} \geq 1/2$. By our assumption by contradiction, take a unitary $w \in \mathcal{U}(B)$ such that

$$\tau(w^* \mathfrak{m}_{\rho_0}(w)) \leq \frac{\varepsilon^2}{8}.$$

It follows that $\|\mathcal{P}_{K_2}(w)\|_2 \leq \varepsilon/2$. By the Kaplansky density theorem, take w_0 in the dense $*$ -subalgebra

$$M_0 = \text{span}\{au_g \mid a \in A, g \in \Gamma\}$$

with $\|w_0\| \leq 1$ and $\|w - w_0\|_2 \leq \varepsilon/2$. Put $x = \mathfrak{m}_\varphi(w_0)$. Then,

$$\|\mathfrak{m}_\varphi(w) - x\|_2 = \|\mathfrak{m}_\varphi(w - w_0)\|_2 \leq \|w - w_0\|_2 \leq \frac{\varepsilon}{2}.$$

Since $\|\mathcal{P}_{K_2}(w)\|_2 \leq \varepsilon/2$, also $\|w - \mathfrak{m}_\varphi(w)\|_2 \leq \varepsilon/2$. So, our element $x \in M$ has the following properties.

- $x \in \text{span}\{au_g \mid a \in A, |g| > 2K_1\}$,
- $\|x\| \leq 3$,
- $\|w - x\|_2 \leq \varepsilon$.

Since B is abelian, $w \in \mathcal{U}(B)$ and v_k normalizes B , we have

$$p = v_k w v_k^* w v_k w^* v_k^* w^*$$

for all k . We now replace v_k by y_k and w by x . We use the estimates for $\|x\|$, $\|y_k\|$, $\|v_k - y_k\|_2$ and $\|w - x\|_2$, to conclude that

$$\begin{aligned} &\|p - y_k x y_k^* x y_k x^* y_k^* x^*\|_2 \\ &\leq \varepsilon(1 + 3 + 2 \cdot 3 + 3 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 2 \cdot 3 + 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3 + 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 3) \\ &= 1036\varepsilon = \frac{1}{2}\tau(p). \end{aligned}$$

It follows that for all k ,

$$|\tau(y_k x y_k^* x y_k x^* y_k^* x^*)| \geq \frac{1}{2}\tau(p).$$

We claim however that the left-hand side tends to 0 when $k \rightarrow \infty$.

Since $x \in \text{span}\{au_g \mid a \in A, |g| > 2K_1\}$, it suffices to prove that

$$\tau(y_k u_g a u_{g'} y_k^* u_h b u_{h'} y_k u_r c u_{r'} y_k^* u_t d u_{t'}) \rightarrow 0 \quad (5.2)$$

for all $a, b, c, d \in A$ and for all group elements $g, g', h, h', k, k', r, r', t, t'$ having length at least K_1 and being chosen such that the concatenations gg', hh', rr' and tt' are reduced.

We now study the expressions

$$u_{t'} y_k u_g, \quad u_{g'} y_k^* u_h, \quad u_{h'} y_k u_r \quad \text{and} \quad u_{r'} y_k^* u_t.$$

According to Lemma 5.5, every of these four expressions can be written as a sum $a_k^{(i)} + b_k^{(i)}$, $i = 1, 2, 3, 4$, of bounded sequences satisfying the following properties: $\|b_k^{(i)}\|_2 \rightarrow 0$ and we have $a_k^{(i)} = \mathcal{P}_{g_0, h_0}(a_k^{(i)})$ where g_0 is the first letter of resp. t', g', h', r' and h_0 is the last letter of resp. g, h, r, t .

Then, the left hand side of (5.2) equals

$$\tau((a_k^{(1)} + b_k^{(1)}) a (a_k^{(2)} + b_k^{(2)}) b (a_k^{(3)} + b_k^{(3)}) c (a_k^{(4)} + b_k^{(4)}) d).$$

Developing all the sums, the one term that only involves $a_k^{(i)}$ equals zero because there is no simplification between consecutive factors of the product and all the other terms tend to zero because $\|b_k^{(i)}\|_2 \rightarrow 0$ and the $(a_k^{(i)})$ are bounded.

We have reached a contradiction and hence we have proven the lemma. \square

It is now easy to prove Theorem 5.6.

Proof of Theorem 5.6. Denote by $(v_s)_{s \in \Lambda}$ the canonical unitaries in $B \rtimes \Lambda$. Put $\varepsilon = \tau(p)/2072$. In the situation where Γ_1 has a non-amenable subgroup H with the relative property (T), we apply Lemma 3.1 to the completely positive maps m_ρ , $\rho \rightarrow 1$ and the von Neumann subalgebra $M_0 = L(H)$. In the situation where Γ_1 has two non-amenable commuting subgroups H_1, H_2 , we apply Lemma 3.2 to the deformation α_t , $t \rightarrow 0$, introduced in Section 5.1 and the commuting subalgebras $L(H_1), L(H_2)$. Combined with (5.1), we always find $0 < \rho < 1$ and a sequence $s_k \in \Lambda$ such that

- $\|v_{s_k} - m_\rho(v_{s_k})\|_2 \leq \varepsilon/2$ for all k .
- $\|E_P(x v_{s_k} y)\|_2 \rightarrow 0$ for all $x, y \in M$.

By Lemma 5.7 we get a $0 < \rho < 1$ and a $\delta > 0$ such that $\tau(w^* m_\rho(w)) \geq \delta$ for all $w \in \mathcal{U}(B)$. By Theorem 5.4 we have that $B \prec_M P$ or that $\mathcal{N}_{pMp}(B)'' \prec_M A \rtimes \Gamma_i$ for some $i = 1, 2$. Since $\mathcal{N}_{pMp}(B)'' = pMp$ and since $\Gamma_i < \Gamma$ has infinite index, the second option is impossible, concluding the proof of the theorem. \square

5.4 Proof of Theorems 5.2 and 1.4

Proof of Theorem 5.2. Put $A = M_n(\mathbb{C}) \otimes L^\infty(X)$ with $\Gamma \curvearrowright A$ acting trivially on $M_n(\mathbb{C})$. In the proof, we do not write the isomorphism π and we put $B = L^\infty(Y)$. So, $A \rtimes \Gamma = M_n(\mathbb{C}) \otimes M$ and p is a projection in $A \rtimes \Gamma$ such that $p(A \rtimes \Gamma)p = B \rtimes \Lambda$. By Theorem 5.6, we get an intertwining bimodule between B and $A \rtimes \Sigma$. So, we also have

$$B \prec_M L^\infty(X) \rtimes \Sigma.$$

By condition 3 in Definition 5.1 and [PV06, Theorem 6.16], it follows that $B \prec_M L^\infty(X)$. Then, [Po01, Theorem A.1] (see also [Va06, Theorem C.3]) provides the conclusion of the theorem. \square

Proof of Theorem 1.4. Since Γ acts ergodically on (X, μ) and Γ is ICC, it follows that M is a factor. Assume that B is a group measure space Cartan subalgebra in $pM^n p$. Repeating the previous proof, it follows that $B \prec_M L^\infty(X)$. By [OP07, Theorem 4.11], it follows that $\Gamma \curvearrowright (X, \mu)$ is free. \square

5.5 II_1 factors with at least two group measure space Cartan subalgebras

By definition, the family \mathcal{G} consists of amalgamated free products $\Gamma = \Gamma_1 *_{\Sigma} \Gamma_2$, where Γ_1 satisfies a rigidity condition (5.1.1), where Σ is amenable and different from Γ_2 (5.1.2) and where $\bigcap_{i=1}^k g_i \Sigma g_i^{-1}$ is finite for some $g_1, \dots, g_k \in \Gamma$ (5.1.3). Without this last condition, it is possible to give examples of group actions such that $M = L^\infty(X) \rtimes \Gamma$ admits at least two Cartan subalgebras that are non-conjugate by an automorphism of M .

Indeed, Connes and Jones [CJ82] provide examples of free ergodic p.m.p. actions $\Gamma \times \Sigma \curvearrowright (X, \mu)$ such that the II_1 factor $L^\infty(X) \rtimes (\Gamma \times \Sigma)$ has at least two group measure space Cartan subalgebras that are non-conjugate by an automorphism. In their construction, Γ can be any non-amenable group and Σ is a specific infinite amenable group. In particular, one can consider

$$(\Gamma_1 * \Gamma_2) \times \Sigma = (\Gamma_1 \times \Sigma) *_{\Sigma} (\Gamma_2 \times \Sigma)$$

and provide examples where conditions 5.1.1 and 5.1.2 are satisfied, but condition 5.1.3 is not.

Mimicking [OP08, Section 7], we give other examples of II_1 factors with at least two group measure space decompositions. The following general non-uniqueness statement is a consequence of Example 5.8. Whenever $\Gamma = H \rtimes G$ is a semi-direct product group with H being infinite abelian, then Γ admits free ergodic p.m.p. actions such that the corresponding group measure space II_1 factor has at least two non unitarily conjugate group measure space Cartan subalgebras. Whenever $G = G_1 *_{\Sigma} G_2$, also $\Gamma = (H \rtimes G_1) *_{H \rtimes \Sigma} (H \rtimes G_2)$. As such, we get again examples where conditions 5.1.1 and 5.1.2 are satisfied, but condition 5.1.3 is not.

Example 5.8. Let H be an infinite abelian group and $G \curvearrowright^\alpha H$ an action by automorphisms. Let $H \hookrightarrow K$ be a dense embedding of H into the compact abelian group K . Assume that $G \curvearrowright H$ extends to an action by homeomorphisms of K that we still denote by α . Whenever $G \curvearrowright (X, \mu)$ is a free ergodic p.m.p. action, consider the free ergodic p.m.p. action

$$H \rtimes G \curvearrowright K \times X \quad \text{given by} \quad \begin{cases} h \cdot (k, x) = (h + k, x) \\ g \cdot (k, x) = (\alpha_g(k), g \cdot x) \end{cases} \quad \text{for all } h \in H, g \in G, k \in K, x \in X.$$

Dualizing the embedding $H \hookrightarrow K$, we get the embedding $\widehat{K} \hookrightarrow \widehat{H}$ and the action of G by automorphisms of \widehat{K} and \widehat{H} . We canonically have

$$L^\infty(K \times X) \rtimes (H \rtimes G) = L^\infty(\widehat{H} \times X) \rtimes (\widehat{K} \rtimes G).$$

First of all, the group measure space Cartan subalgebras $L^\infty(K \times X)$ and $L^\infty(\widehat{H} \times X)$ are never *unitarily conjugate*. Indeed, if $h_n \in H$ is a sequence tending to infinity in H , the unitaries $u_{h_n} \in L(H)$ satisfy $\|E_{L^\infty(K \times X)}(au_{h_n}b)\|_2 \rightarrow 0$ for all $a, b \in L^\infty(K \times X) \rtimes (H \rtimes G)$. It follows that $L^\infty(\widehat{H}) = L(H) \not\prec L^\infty(K \times X)$. A fortiori, $L^\infty(\widehat{H} \times X)$ cannot be unitarily conjugated onto $L^\infty(K \times X)$.

In certain examples, the Cartan subalgebras $L^\infty(K \times X)$ and $L^\infty(\widehat{H} \times X)$ are *conjugate by an automorphism*. This is, for instance, always the case when the action $G \curvearrowright H$ is trivial. Then, the crossed product II_1 factor is the tensor product of $L^\infty(X) \rtimes G$ and $L^\infty(K) \rtimes H = L^\infty(\widehat{H}) \rtimes \widehat{K}$. The second tensor factor is the hyperfinite II_1 factor and hence, the Cartan subalgebras $L^\infty(K)$ and $L^\infty(\widehat{H})$ are conjugate by an automorphism [OW80, CFW81].

In other examples, the Cartan subalgebras $L^\infty(K \times X)$ and $L^\infty(\widehat{H} \times X)$ are *non conjugate by an automorphism*. Consider $G = \text{SL}(n, \mathbb{Z}) \curvearrowright H = \mathbb{Z}^n$ and $\mathbb{Z}^n \hookrightarrow K = \mathbb{Z}_p^n$, where \mathbb{Z}_p denotes the ring of p -adic integers for some prime number p . It follows that $\widehat{K} \rtimes G$ is the direct limit of a sequence of groups that are virtually isomorphic with $\text{SL}(n, \mathbb{Z})$.

- For $n = 2$, $H \rtimes G$ does not have the Haagerup property (because H is an infinite subgroup with the relative property (T)), while $\widehat{K} \rtimes G$ has the Haagerup property (as the direct limit of groups with the Haagerup property).
- For $n = 3$, $H \rtimes G$ has property (T), while $\widehat{K} \rtimes G$ does not have property (T) (as the direct limit of a strictly increasing sequence of groups).

Since both property (T) [Fu99a, Corollary 1.4] and the Haagerup property [Po01, Remark 3.5.6°] are measure equivalence invariants, it follows that for $n = 2, 3$, the group actions $(H \rtimes G) \curvearrowright (K \times X)$ and $(\widehat{K} \rtimes G) \curvearrowright (\widehat{H} \times X)$ are not stably orbit equivalent. Hence, the corresponding group measure space Cartan subalgebras are not conjugate by an automorphism either.

5.6 Amalgamated free products over groups with the Haagerup property

We mention that the result in Theorem 5.2 also holds for certain amalgamated free products $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ over *non-amenable groups Σ with the Haagerup property*, once the free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is such that $L^\infty(X) \rtimes \Sigma$ still has the Haagerup property. This latter condition is not automatic, but holds for plain Bernoulli actions by [CSV09, Theorem 1.1]. More precisely, apart from the Haagerup property of $L^\infty(X) \rtimes \Sigma$, one has to assume that $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ is such that Γ_1 admits an infinite subgroup with property (T), that Σ has the Haagerup property, that $\Sigma \neq \Gamma_2$ and that the ‘malnormality’ condition 5.1.3 holds.

In order to prove such a statement, it suffices to observe that Lemma 3.1 still holds when M_0 has property (T) and P has the Haagerup property. In the proof of Theorem 5.2, Lemma 3.1 is applied to $P = L^\infty(X) \rtimes \Sigma$.

6 Stable W^* -superrigidity theorems

6.1 W^* -superrigidity

Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action. Denote $A = L^\infty(X)$ and denote by $(\sigma_g)_{g \in \Gamma}$ the corresponding group of automorphisms of A . We denote by $Z^1(\Gamma \curvearrowright X)$ the abelian group of scalar 1-cocycles for the action $\Gamma \curvearrowright X$, i.e. the group of functions $\omega : \Gamma \rightarrow \mathcal{U}(A) : g \mapsto \omega_g$ satisfying $\omega_{gh} = \omega_g \sigma_g(\omega_h)$ for all $g, h \in \Gamma$.

Let $\Delta : X \rightarrow Y$ be a conjugacy between the free ergodic p.m.p. actions $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$, with corresponding group isomorphism $\delta : \Gamma \rightarrow \Lambda$. Define the isomorphism $\Delta_* : L^\infty(X) \rightarrow L^\infty(Y) : \Delta(a) = a \circ \Delta^{-1}$. Whenever $\omega \in Z^1(\Gamma \curvearrowright A)$, we get an isomorphism

$$\theta : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda : \theta(au_g) = \Delta_*(a\omega_g) u_{\delta(g)} \quad \text{for all } a \in L^\infty(X), g \in \Gamma. \quad (6.1)$$

Definition 6.1. We call a free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ *W^* -superrigid* if the following property holds. If $\Lambda \curvearrowright (Y, \eta)$ is a free ergodic p.m.p. action and $\pi : L^\infty(X) \rtimes \Gamma \rightarrow L^\infty(Y) \rtimes \Lambda$ is an isomorphism, then the groups Γ and Λ are isomorphic, their actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ are conjugate and, up to a unitary conjugacy, π is of the form (6.1).

Actually, the notion of *stable W^* -superrigidity* is more natural (see Definition 6.4 below), allowing in the correct way for amplifications. Indeed, in order for $\Gamma \curvearrowright (X, \mu)$ to be W^* -superrigid in the above sense, Γ should not have finite normal subgroups, which is a somewhat restrictive assumption.

We start with the following more precise version of Theorem 1.2.

Theorem 6.2. *Let $n \geq 3$ and denote by T_n the subgroup of upper triangular matrices in $\text{PSL}(n, \mathbb{Z})$. Put $\Gamma = \text{PSL}(n, \mathbb{Z}) *_{T_n} \text{PSL}(n, \mathbb{Z})$. Then, every free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ with the property that all finite index subgroups of T_n act ergodically on (X, μ) , is W^* -superrigid. In particular, all free p.m.p. mixing actions of Γ are W^* -superrigid.*

Proof. Take a free p.m.p. action $\Gamma \curvearrowright (X, \mu)$ such that the restriction to every finite index subgroup of T_n is ergodic. Assume that $L^\infty(X) \rtimes \Gamma = L^\infty(Y) \rtimes \Lambda$ for some free ergodic p.m.p. action $\Lambda \curvearrowright (Y, \eta)$. By Theorem 5.2, we can unitarily conjugate $L^\infty(Y)$ onto $L^\infty(X)$. Hence, $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ are orbit equivalent. But then, [Ki09, Theorem 1.4] yields the conclusion of the theorem. \square

6.2 Stable W^* -superrigidity

In order to introduce the more natural notion of *stable W^* -superrigidity*, we first discuss stable isomorphism of II_1 factors. If M is a II_1 factor and $t > 0$, the amplification M^t is defined as $M^t := p(M_n(\mathbb{C}) \otimes M)p$, where $p \in M_n(\mathbb{C}) \otimes M$ is a projection satisfying $(\text{Tr} \otimes \tau)(p) = t$. Note that M^t is uniquely defined up to unitary conjugacy.

Definition 6.3. Let M and N be II_1 factors. A *stable isomorphism* between M and N is an isomorphism $\pi : N \rightarrow M^t$ for some $t > 0$.

Given the isomorphism $\pi : N \rightarrow M^t$ and a projection $p \in M_n(\mathbb{C}) \otimes M$ with $(\text{Tr} \otimes \tau)(p) = t$, define the Hilbert space $\mathcal{H}^\pi := (M_{1,n}(\mathbb{C}) \otimes L^2(M))p$. The formula $a \cdot \xi \cdot b := a\xi\pi(b)$ turns \mathcal{H}^π into an M - N -bimodule such that the right N -action equals the commutant of the left M -action and vice versa.

We equivalently define a *stable isomorphism* between M and N as being an M - N -bimodule ${}_M\mathcal{H}_N$ such that the right N -action equals the commutant of the left M -action (and, equivalently, vice versa). Every stable isomorphism ${}_M\mathcal{H}_N$ is unitarily equivalent with ${}_M\mathcal{H}^\pi_N$ for an isomorphism $\pi : N \rightarrow M^t$ that is uniquely determined up to unitary conjugacy.

We call the number $t > 0$ the *compression constant* of the stable isomorphism ${}_M\mathcal{H}^\pi_N$.

Stable isomorphisms can be composed using the Connes tensor product. Of course, we have

$$\mathcal{H}^\pi \otimes_N \mathcal{H}^\eta \cong \mathcal{H}^{(\text{id} \otimes \pi)\eta}.$$

Fix free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$. Stabilizing the notions of W^* -equivalence, orbit equivalence and conjugacy (see the beginning of Section 2), we introduce the following terminology.

First recall that a free p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is said to be *induced* from $\Gamma_0 \curvearrowright X_0$ if $\Gamma_0 < \Gamma$ is a finite index subgroup, $X_0 \subset X$ is a non-negligible Γ_0 -invariant subset and, up to measure zero, the sets $g \cdot X_0$, $g \in \Gamma/\Gamma_0$, form a partition of X .

- We call *stable W^* -equivalence* between the actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$, any stable isomorphism between $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$.
- We call *stable orbit equivalence* between the actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$, any stable isomorphism between the orbit equivalence relations $\mathcal{R}(\Gamma \curvearrowright X)$ and $\mathcal{R}(\Lambda \curvearrowright Y)$, i.e. any isomorphism $\Delta : X_1 \rightarrow Y_1$ between non-negligible subsets $X_1 \subset X$, $Y_1 \subset Y$ satisfying

$$\Delta(X_1 \cap \Gamma \cdot x) = Y_1 \cap \Lambda \cdot \Delta(x)$$

for a.e. $x \in X_1$. We call $\mu(X_1)/\eta(Y_1)$ the compression constant of the stable orbit equivalence.

- We call *stable conjugacy* between the actions $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$, any conjugacy between the actions $\frac{\Gamma_0}{G} \curvearrowright \frac{X_0}{G}$ and $\frac{\Lambda_0}{H} \curvearrowright \frac{Y_0}{H}$ where $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ are induced from $\Gamma_0 \curvearrowright X_0$, $\Lambda_0 \curvearrowright Y_0$ and where $G < \Gamma_0$, $H < \Lambda_0$ are finite normal subgroups. The number $\frac{|H|}{|G|} \frac{[\Lambda : \Lambda_0]}{[\Gamma : \Gamma_0]}$ is called the compression constant of the stable conjugacy.

From stable conjugacy to stable orbit equivalence. If $\Gamma \curvearrowright (X, \mu)$ is induced from $\Gamma_0 \curvearrowright X_0$ and if $G < \Gamma_0$ is a finite normal subgroup, the canonical stable orbit equivalence between $\Gamma \curvearrowright (X, \mu)$ and $\frac{\Gamma_0}{G} \curvearrowright \frac{X_0}{G}$ is defined by taking a fundamental domain $X_1 \subset X_0$ for the action $G \curvearrowright X_0$ and restricting the quotient map $X_0 \rightarrow \frac{X_0}{G}$ to X_1 . The compression constant is $(|G|[\Gamma : \Gamma_0])^{-1}$. The stable orbit equivalence associated with a stable conjugacy between $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ is defined as the composition of the canonical stable orbit equivalences with $\frac{\Gamma_0}{G} \curvearrowright \frac{X_0}{G}$, resp. $\frac{\Lambda_0}{H} \curvearrowright \frac{Y_0}{H}$, together with the conjugacy between both actions.

From stable orbit equivalence to stable W^ -equivalence.* Let $\Delta : X_1 \rightarrow Y_1$ be a stable orbit equivalence and denote by $p \in L^\infty(X)$, $q \in L^\infty(Y)$ the projections with support X_1, Y_1 . Then Δ gives rise to a canonical isomorphism $\pi_\Delta : q(L^\infty(Y) \rtimes \Lambda)q \rightarrow p(L^\infty(X) \rtimes \Gamma)p$ satisfying $\pi_\Delta(b) = b \circ \Delta$ for all $b \in L^\infty(Y_1)$. The isomorphism π_Δ amplifies to a stable W^* -equivalence $L^\infty(Y) \rtimes \Lambda \rightarrow (L^\infty(X) \rtimes \Gamma)^t$, with $t = \mu(X_1)/\eta(Y_1)$, that we still denote by π_Δ .

Definition 6.4. A free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ is said to be *stably W^* -superrigid* if the following holds. Whenever π is a stable W^* -equivalence between $\Gamma \curvearrowright (X, \mu)$ and an arbitrary free ergodic p.m.p. action $\Lambda \curvearrowright (Y, \eta)$, it follows that the actions are stably conjugate and that π equals the composition of

- the canonical stable W^* -equivalence given by the stable conjugacy,
- the automorphism of $L^\infty(X) \rtimes \Gamma$ given by an element of $Z^1(\Gamma \curvearrowright X)$,
- an inner automorphism of $L^\infty(X) \rtimes \Gamma$.

Let $\Gamma \curvearrowright (X, \mu)$ be stably W^* -superrigid. If moreover Γ has no finite normal subgroups and if finite index subgroups of Γ still act ergodically on (X, μ) , then $\Gamma \curvearrowright (X, \mu)$ is W^* -superrigid in the sense of Section 1.

For certain families of group actions $\Gamma \curvearrowright (X, \mu)$, all 1-cocycles with values in S^1 are known to be cohomologous to a group morphism and then we may assume that the corresponding automorphism of $L^\infty(X) \rtimes \Gamma$ is implemented by a character $\Gamma \rightarrow S^1$.

From stable W^ -equivalence to stable orbit equivalence.* Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action satisfying the conclusion of Theorem 5.2. Whenever π is a stable W^* -equivalence between $\Gamma \curvearrowright (X, \mu)$ and an arbitrary free ergodic p.m.p. action $\Lambda \curvearrowright (Y, \eta)$, we then find a stable orbit equivalence Δ between $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ such that π equals the composition of π_Δ , the automorphism of $L^\infty(X) \rtimes \Gamma$ given by an element of $Z^1(\Gamma \curvearrowright X)$ and an inner automorphism.

From stable orbit equivalence to stable conjugacy. Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action that is cocycle superrigid with arbitrary countable target groups (see paragraph 2.4 for terminology). Whenever Δ is a stable orbit equivalence between $\Gamma \curvearrowright (X, \mu)$ and an arbitrary free ergodic p.m.p. action $\Lambda \curvearrowright (Y, \eta)$, it follows (see [Po05, Proposition 5.11], [Va06, Lemma 4.7]) that the actions are stably conjugate and that Δ equals the composition of the canonical stable orbit equivalence given by the stable conjugacy and an inner automorphism, i.e. an automorphism Δ_0 of (X, μ) satisfying $\Delta_0(x) \in \Gamma \cdot x$ for a.e. $x \in X$. Even more so, it actually follows that there exists a finite normal subgroup $G \triangleleft \Gamma$ and that $\Lambda \curvearrowright Y$ is induced from $\Lambda_0 \curvearrowright Y_0$ such that the actions $\frac{\Gamma}{G} \curvearrowright \frac{X}{G}$ and $\Lambda_0 \curvearrowright Y_0$ are conjugate.

Summarizing the previous two paragraphs, we have proven the following lemma.

Lemma 6.5. *Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action that satisfies the conclusion of Theorem 5.2 and that is cocycle superrigid with arbitrary countable target groups, as well as with target group S^1 . Then, $\Gamma \curvearrowright (X, \mu)$ is stably W^* -superrigid. Even more precisely, we have the following.*

Let $\Lambda \curvearrowright (Y, \eta)$ be an arbitrary free ergodic p.m.p. action and $\pi : L^\infty(X) \rtimes \Gamma \rightarrow (L^\infty(Y) \rtimes \Lambda)^t$ a $$ -isomorphism for some $t > 0$. Then, $\Lambda \curvearrowright Y$ is induced from $\Lambda_0 \curvearrowright Y_0$ and there exist*

- *a finite normal subgroup $G \triangleleft \Gamma$ and a group isomorphism $\delta : \frac{\Gamma}{G} \rightarrow \Lambda_0$,*
- *a measure space isomorphism $\Delta : \frac{X}{G} \rightarrow Y_0$ conjugating the actions, i.e. $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ for all $g \in \frac{\Gamma}{G}$ and a.e. $x \in X$,*
- *a character $\omega : \Gamma \rightarrow S^1$,*

such that $t = \frac{|G|}{[\Lambda : \Lambda_0]}$ and such that, after a unitary conjugacy, π equals the composition of

- *the automorphism π_ω of $L^\infty(X) \rtimes \Gamma$ given by $\pi_\omega(au_g) = \omega(g)au_g$,*
- *the canonical isomorphism $L^\infty(X) \rtimes \Gamma \rightarrow (L^\infty(\frac{X}{G}) \rtimes \frac{\Gamma}{G})^n$ where $n = |G|$,*
- *the isomorphism $\pi_\Delta : L^\infty(\frac{X}{G}) \rtimes \frac{\Gamma}{G} \rightarrow L^\infty(Y_0) \rtimes \Lambda_0$ given by $\pi_\Delta(au_g) = (a \circ \Delta^{-1})u_{\delta(g)}$,*

- the canonical isomorphism $L^\infty(Y_0) \rtimes \Lambda_0 \rightarrow (L^\infty(Y) \rtimes \Lambda)^{\frac{1}{m}}$ where $m = [\Lambda : \Lambda_0]$.

Remark 6.6. Let $\Gamma \curvearrowright (X, \mu)$ be a stably W^* -superrigid, free ergodic p.m.p. action. If Γ has no finite normal subgroups and if the finite index subgroups of Γ act ergodically on (X, μ) , then $L^\infty(X) \rtimes \Gamma$ has trivial fundamental group.

Recall that given any action $\Gamma \curvearrowright I$ of a countable group Γ on a countable set I , the *generalized Bernoulli action* with base probability space (X_0, μ_0) , is defined as $\Gamma \curvearrowright (X_0, \mu_0)^I$ where $(g \cdot x)_i = x_{g^{-1} \cdot i}$ for all $g \in \Gamma$, $x \in X_0^I$ and $i \in I$.

Recall that given any orthogonal representation $\pi : \Gamma \rightarrow O(\mathcal{K}_{\mathbb{R}})$ of Γ on the real Hilbert space $\mathcal{K}_{\mathbb{R}}$, the Gaussian functor allows to define the *Gaussian* p.m.p. action of Γ on the Gaussian probability space defined by $\mathcal{K}_{\mathbb{R}}$.

Theorem 6.7. Let Γ_1, Γ_2 be countable groups with a common infinite amenable subgroup Σ . Assume that Σ is a proper, normal subgroup of Γ_2 and that there exist $g_1, \dots, g_k \in \Gamma_1$ such that

$$\bigcap_{i=1}^k g_i \Sigma g_i^{-1} \text{ is finite.}$$

Put $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$.

1. If Γ_1 admits a non-amenable normal subgroup H with the relative property (T), then all of the following actions are stably W^* -superrigid.
 - Every free p.m.p. action $\Gamma \curvearrowright (X, \mu)$ whose restriction to Γ_1 is a generalized Bernoulli action $\Gamma_1 \curvearrowright (X_0, \mu_0)^I$ with the property that both $H \cdot i$ and $\Sigma \cdot i$ are infinite for all $i \in I$.
 - Every free p.m.p. action $\Gamma \curvearrowright (X, \mu)$ whose restriction to Γ_1 is a Gaussian action defined by an orthogonal representation $\pi : \Gamma_1 \rightarrow O(\mathcal{K}_{\mathbb{R}})$ with the property that both restrictions $\pi|_H$ and $\pi|_\Sigma$ have no non-zero finite dimensional subrepresentations.
2. Suppose that Γ_1 admits non-amenable commuting subgroups H and H' such that H is normal in Γ_1 . If $\Gamma \curvearrowright (X, \mu)$ is a free p.m.p. action whose restriction to Γ_1 is a generalized Bernoulli action $\Gamma_1 \curvearrowright (X_0, \mu_0)^I$ with the properties that $\text{Stab } i \cap H'$ is amenable for all $i \in I$ and that both $H \cdot i$ and $\Sigma \cdot i$ are infinite for all $i \in I$, then $\Gamma \curvearrowright (X, \mu)$ is stably W^* -superrigid.

In particular, for all groups Γ mentioned in this theorem, the plain Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ is stably W^* -superrigid.

More precisely, all actions $\Gamma \curvearrowright X$ appearing in the theorem satisfy the conclusions of Lemma 6.5.

Before proving Theorem 6.7, we provide the following concrete examples of stably W^* -superrigid group actions.

Example 6.8. 1. Denote by $T_n < \text{PSL}(n, \mathbb{Z})$ the subgroup of upper triangular matrices and assume $n \geq 3$. For an arbitrary non-trivial group Λ and an arbitrary infinite subgroup $\Sigma < T_n$, put $\Gamma = \text{PSL}(n, \mathbb{Z}) *_\Sigma (\Sigma \times \Lambda)$. Whenever $\Gamma \curvearrowright I$ is such that $\Sigma \cdot i$ is infinite for all $i \in I$, the generalized Bernoulli actions $\Gamma \curvearrowright (X_0, \mu_0)^I$ are stably W^* -superrigid.

2. Let Σ be an infinite amenable subgroup of the non-amenable group H . Assume that H is finitely generated and that $\Sigma \cap \mathcal{Z}(H) = \{e\}$. Let Λ be an arbitrary non-trivial group. Define $\Gamma_1 = H \times H$, view Σ as a subgroup of Γ_1 diagonally and put $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$. Whenever $\Gamma \curvearrowright I$ is such that $\Sigma \cdot i$ and $(H \times \{e\}) \cdot i$ is infinite for all $i \in I$, the generalized Bernoulli actions $\Gamma \curvearrowright (X_0, \mu_0)^I$ are stably W^* -superrigid.

Actually, exploiting the full strength of Theorem 6.7, many more examples of stably W^* -superrigid actions can be given. Given an arbitrary faithful⁽⁵⁾ p.m.p. action $\Lambda \curvearrowright (X_0, \mu_0)$, we construct a stably W^* -superrigid, free ergodic p.m.p. action $\Gamma \curvearrowright (X, \mu)$ such that Λ is a subgroup of Γ and such that $\Lambda \curvearrowright (X_0, \mu_0)$ is a quotient of the restriction of $\Gamma \curvearrowright (X, \mu)$ to Λ .

Recall first the construction of a *co-induced action*. Let $\Lambda < \Gamma$ be a subgroup and $\Lambda \curvearrowright (X_0, \mu_0)$ a p.m.p. action. Choose a map $\pi : \Gamma \rightarrow \Lambda$ satisfying $\pi(gh) = \pi(g)h$ for all $g \in \Gamma, h \in \Lambda$. Define the 1-cocycle ω for the action $\Gamma \curvearrowright \Gamma/\Lambda$ with values in Λ by the formula

$$\omega(g, h\Lambda) = \pi(gh)\pi(h)^{-1}.$$

Different choices of π lead to cohomologous 1-cocycles ω . Define the probability space $(X, \mu) := (X_0, \mu_0)^{\Gamma/\Lambda}$ and the p.m.p. action $\Gamma \curvearrowright (X, \mu)$ given by

$$(g^{-1} \cdot x)_{h\Lambda} := \omega(g, h\Lambda)^{-1} \cdot x_{gh\Lambda} \quad \text{for all } g, h \in \Lambda, x \in X.$$

We call $\Gamma \curvearrowright (X, \mu)$ the *co-induced action of $\Lambda \curvearrowright (X_0, \mu_0)$ to Γ* . Different choices of π lead to conjugate actions.

Note that we can choose π such that $\pi(h) = h$ for all $h \in \Lambda$. Then, the quotient map $\theta : X \rightarrow X_0 : x \mapsto x_{e\Lambda}$ satisfies $\theta(h \cdot x) = h \cdot \theta(x)$ for all $h \in \Lambda, x \in X$. Hence, $\Lambda \curvearrowright X_0$ arises as a quotient of the restriction of $\Gamma \curvearrowright X$ to Λ .

Example 6.9. Take $\Gamma = \text{PSL}(n, \mathbb{Z}) *_{\Sigma} (\Sigma \times \Lambda)$ as in Example 6.8.1 or take $\Gamma = (H \times H) *_{\Sigma} (\Sigma \times \Lambda)$ as in Example 6.8.2. Let $\Lambda \curvearrowright (X_0, \mu_0)$ be an arbitrary faithful p.m.p. action. Then, the co-induced action $\Gamma \curvearrowright (X, \mu)$ is stably W^* -superrigid.

In the first example, put $\Gamma_1 = \text{PSL}(n, \mathbb{Z})$ and in the second example, put $\Gamma_1 = H \times H$. In order to apply Theorem 6.7, it suffices to observe that the restriction of the co-induced action $\Gamma \curvearrowright (X, \mu)$ to Γ_1 is a Γ_1 -Bernoulli action. Indeed, the action $\Gamma_1 \times \Lambda \curvearrowright \Gamma$ by left-right multiplication is free. Therefore, we can choose $\pi : \Gamma \rightarrow \Lambda$ satisfying $\pi(ghk) = \pi(h)k$ for all $g \in \Gamma_1, h \in \Gamma, k \in \Lambda$. Associated with π is the 1-cocycle ω for $\Gamma \curvearrowright \Gamma/\Lambda$, which now satisfies $\omega(g, h\Lambda) = e$ for all $g \in \Gamma_1, h \in \Gamma$. Hence, the restriction $\Gamma_1 \curvearrowright (X, \mu)$ is precisely the Bernoulli action $\Gamma_1 \curvearrowright (X_0, \mu_0)^{\Gamma/\Lambda}$. The latter can be seen as the plain Bernoulli action $\Gamma_1 \curvearrowright Y_0^{\Gamma_1}$, where $Y_0 := X_0^{\Gamma_1 \backslash \Gamma/\Lambda}$.

Example 6.10. Let Γ_1 be an infinite group with property (T), $\Sigma < \Gamma_1$ an infinite subgroup with the Haagerup property and Λ an arbitrary non-trivial group. Put $\Gamma = \Gamma_1 *_{\Sigma} (\Sigma \times \Lambda)$. Using paragraph 5.6 instead of Theorem 5.2, it follows that the plain Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^{\Gamma}$ is stably W^* -superrigid. Similarly (cf. Example 6.9), the co-induced action $\Gamma \curvearrowright (X, \mu)$ of an arbitrary faithful p.m.p. action $\Lambda \curvearrowright (X_0, \mu_0)$, follows stably W^* -superrigid.

6.3 Cocycle superrigidity for actions of amalgamated free products

Recall from paragraph 2.4 the notion of cocycle superrigidity. Theorem 6.7 will be proven as a consequence of Theorem 5.2 and a number of cocycle superrigidity theorems from [Po05, Po06a] that we recall here.

We first prove the following permanence lemma for \mathcal{L} -cocycle superrigidity. It is a direct consequence of techniques in [Po05, Section 3], but we give a short and full proof for the convenience of the reader. We also deal with arbitrary finite index subgroups, which will be useful in Section 7.

⁽⁵⁾This means that non-trivial group elements act non-trivially.

Lemma 6.11. *Let $\Gamma = \Gamma_1 *_{\Sigma} \Gamma_2$ be an amalgamated free product where $\Sigma \triangleleft \Gamma_2$ is a normal subgroup. Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action and assume that its restriction $\Sigma \curvearrowright (X, \mu)$ is weakly mixing. Let \mathcal{L} be a Polish group whose topology is induced by a separable complete bi-invariant metric (for instance, $\mathcal{L} \in \mathcal{U}_{\text{fin}}$).*

If for every finite index subgroup $\Gamma'_1 < \Gamma_1$ the action $\Gamma'_1 \curvearrowright (X, \mu)$ is \mathcal{L} -cocycle superrigid, then $\Gamma' \curvearrowright (X, \mu)$ is \mathcal{L} -cocycle superrigid for all finite index subgroups $\Gamma' < \Gamma$.

Proof. Whenever $\omega : \Gamma \times X \rightarrow \mathcal{L}$ is a 1-cocycle and $\Gamma_0 < \Gamma$ is a subgroup, we say that ω is a group morphism on Γ_0 if for all $g \in \Gamma_0$, the function $x \mapsto \omega(g, x)$ is constant a.e. Equivalently, there exists a group morphism $\delta : \Gamma_0 \rightarrow \mathcal{L}$ such that $\omega(g, x) = \delta(g)$ for all $g \in \Gamma_0$ and a.e. $x \in X$. We similarly say that ω is cohomologous to a group morphism on Γ_0 if ω is cohomologous to a 1-cocycle ω' such that ω' is a group morphism on Γ_0 .

From [Po05, Section 3] we deduce the following two principles. Let $\Lambda \curvearrowright (X, \mu)$ a p.m.p. action and $\Lambda_0 < \Lambda$ a subgroup such that $\Lambda_0 \curvearrowright (X, \mu)$ is weakly mixing. Let $\omega : \Lambda \times X \rightarrow \mathcal{L}$ be a 1-cocycle such that $\omega(h, x) = \delta(h)$ for all $h \in \Lambda_0$, where $\delta : \Lambda_0 \rightarrow \mathcal{L}$ is a group morphism.

Principle 1. If Λ_0 is normal in Λ , then ω is a group morphism on the whole of Λ . Indeed, choose $g \in \Lambda$ and put $\psi(x) = \omega(g, x)$. We have to prove that ψ is essentially constant. But $\psi(h \cdot x) = \delta(hg h^{-1}) \psi(x) \delta(h)^{-1}$ for all $h \in \Lambda_0$ and a.e. $x \in X$. By [PV08b, Lemma 5.4] the function ψ is essentially constant.

Principle 2. If ω is cohomologous to a group morphism on Λ , then ω is already a group morphism on Λ . Indeed, by assumption we find a measurable map $\psi : X \rightarrow \mathcal{L}$ and a group morphism $\rho : \Lambda \rightarrow \mathcal{L}$ such that $\omega(g, x) = \psi(g \cdot x)^{-1} \rho(g) \psi(x)$ for all $g \in \Lambda$ and a.e. $x \in X$. It suffices to prove that ψ is essentially constant. But $\psi(h \cdot x) = \rho(h) \psi(x) \delta(h)^{-1}$ for all $h \in \Lambda_0$ and a.e. $x \in X$. Again by [PV08b, Lemma 5.4] the function ψ is essentially constant.

Take a finite index subgroup $\Gamma' < \Gamma$ and a 1-cocycle $\omega : \Gamma' \times X \rightarrow \mathcal{L}$. We have to prove that ω is cohomologous to a group morphism. Since $\Gamma_1 \cap \Gamma'$ is a finite index subgroup of Γ_1 , we may assume that ω is already a group morphism on $\Gamma_1 \cap \Gamma'$. We prove that ω is then a group morphism on the whole of Γ' .

Take a finite index subgroup $\Gamma'' < \Gamma'$ such that Γ'' is normal in Γ . Define $\Gamma'_i = \Gamma_i \cap \Gamma''$ and $\Sigma' = \Sigma \cap \Gamma''$. Note that $\Sigma' < \Sigma$ has finite index and that $\Gamma'_i \triangleleft \Gamma_i$, $\Sigma' \triangleleft \Gamma'_2$ are normal subgroups. Also $\Sigma' \curvearrowright (X, \mu)$ is still weakly mixing.

We prove by induction on $|g|$ that ω is a group morphism on $g\Gamma'_i g^{-1}$ for all $g \in \Gamma$ and $i = 1, 2$. For $|g| = 0$ we already know that ω is a group morphism on Γ'_1 . In particular, ω is a group morphism on Σ' and by principle 1, also on Γ'_2 . Assume that the statement is true for all elements of length $n - 1$. Take $g \in \Gamma$ with $|g| = n$. Write $g = g_0 h$ where $|g_0| = n - 1$ and $h \in \Gamma_1 \cup \Gamma_2$. If $h \in \Gamma_1$, we have $g\Gamma'_1 g^{-1} = g_0 \Gamma'_1 g_0^{-1}$. By the induction hypothesis, ω is a group morphism on $g\Gamma'_1 g^{-1}$ and, in particular, on $g\Sigma' g^{-1}$. By principle 1, ω is also a group morphism on $g\Gamma'_2 g^{-1}$. Next, if $h \in \Gamma_2$, we have $g\Gamma'_2 g^{-1} = g_0 \Gamma'_2 g_0^{-1}$. By the induction hypothesis, ω is a group morphism on $g\Gamma'_2 g^{-1}$ and, in particular, on $g\Sigma' g^{-1}$. Consider the 1-cocycle $\eta : \Gamma'_1 \times X \rightarrow \mathcal{L}$ given by $\eta(h, x) = \omega(hg h^{-1}, g \cdot x)$. Since $\Gamma'_1 \curvearrowright (X, \mu)$ is \mathcal{L} -cocycle superrigid, it follows that η is cohomologous to a group morphism. But η is already a group morphism on Σ' . By principle 2, η is a group morphism on Γ'_1 and hence, ω is a group morphism on $g\Gamma'_1 g^{-1}$.

Define the subgroup $\Gamma''' < \Gamma'$ generated by $g\Gamma'_i g^{-1}$ for all $g \in \Gamma$ and $i = 1, 2$. We have already proven that ω is a group morphism on Γ''' . By construction, Γ''' is normal in Γ' . So, by principle 1, ω is a group morphism on Γ' . \square

Theorem 6.12. *Let $\Gamma = \Gamma_1 *_\Sigma \Gamma_2$ be an amalgamated free product over an infinite subgroup Σ that is normal in Γ_2 . Let $\Gamma \curvearrowright (X, \mu)$ be a free ergodic p.m.p. action. Assume that the group Γ and its action $\Gamma \curvearrowright (X, \mu)$ satisfy condition 1 or condition 2 in Theorem 6.7. Then, for every finite index subgroup $\Gamma' < \Gamma$, the action $\Gamma' \curvearrowright (X, \mu)$ is \mathcal{U}_{fin} -cocycle superrigid.*

Proof. Both conditions 1 and 2 imply that $\Sigma \curvearrowright (X, \mu)$ is weakly mixing. By Lemma 6.11, it suffices to prove that $\Gamma'_1 \curvearrowright (X, \mu)$ is \mathcal{U}_{fin} -cocycle superrigid for every finite index subgroup $\Gamma'_1 < \Gamma_1$. Under conditions 1, this is a consequence of [Po05, Theorem 0.1]. Under conditions 2, this follows from [Po06a, Theorem 1.1]. \square

Finally, for later use in Section 7 we record the following lemma about weak 1-cocycles (cf. [PS03, Theorem 4.1]).

Lemma 6.13. *Let $\Gamma \curvearrowright (X, \mu)$ be a p.m.p. action that is \mathcal{U}_{fin} -cocycle superrigid. Assume that $\Omega : \Gamma \times \Gamma \rightarrow S^1$ is a scalar 2-cocycle on Γ and that $\omega : \Gamma \times X \rightarrow S^1$ is a measurable map satisfying*

$$\omega(gh, x) = \Omega(g, h)\omega(g, h \cdot x)\omega(h, x)$$

for all $g, h \in \Gamma$ and a.e. $x \in X$. Then, there exist a measurable function $\varphi : X \rightarrow S^1$ and a map $\delta : \Gamma \rightarrow S^1$ such that

$$\omega(g, x) = \varphi(g \cdot x)\delta(g)\overline{\varphi(x)}$$

for all $g \in \Gamma$ and a.e. $x \in X$. In particular, Ω is a coboundary: $\Omega(g, h) = \delta(gh)\overline{\delta(g)\delta(h)}$ for all $g, h \in \Gamma$.

Proof. Form the twisted group von Neumann algebra $N := L_\Omega(\Gamma)$ generated by unitaries $u_g, g \in \Gamma$ satisfying $u_g u_h = \Omega(g, h)u_{gh}$ and equipped with a trace τ satisfying $\tau(u_g) = 0$ if $g \neq e$. Denote by \mathcal{L} the closed subgroup of $\mathcal{U}(N)$ consisting of the unitaries λu_g with $\lambda \in S^1, g \in \Gamma$. Writing $\mu(g, x) = \omega(g, x)u_g$, it follows that μ is a 1-cocycle for $\Gamma \curvearrowright (X, \mu)$ with values in \mathcal{L} .

Since $\Gamma \curvearrowright (X, \mu)$ is \mathcal{U}_{fin} -cocycle superrigid, we find a measurable map $\theta : X \rightarrow \mathcal{L}$ and a group morphism $\eta : \Gamma \rightarrow \mathcal{L}$ such that $\mu(g, x) = \theta(g \cdot x)\eta(g)\theta(x)^{-1}$ for all $g \in \Gamma$ and a.e. $x \in X$. Denoting by $\varphi(x)$, resp. $\delta(g)$, the S^1 -part of $\theta(x)$, resp. $\eta(g)$, the lemma is proven. \square

6.4 Proof of Theorem 6.7

Proof of Theorem 6.7. Since Γ belongs to \mathcal{G} the conclusion of Theorem 5.2 holds. By Theorem 6.12, $\Gamma \curvearrowright (X, \mu)$ is \mathcal{U}_{fin} -cocycle superrigid. In particular, $\Gamma \curvearrowright (X, \mu)$ is cocycle superrigid with arbitrary countable target groups and with target group S^1 . So, the conclusions follow from Lemma 6.5. \square

6.5 Strong rigidity for group von Neumann algebras

Following up Connes' rigidity conjecture [Co80b], Jones asked in [Jo00] whether every isomorphism between property (T) group von Neumann algebras $L(G_1)$ and $L(G_2)$ essentially comes from an isomorphism between the groups G_1 and G_2 (cf. [Po06b, Statement 3.2']). In [PV06, Theorem 7.13], we provided a family of (generalized) wreath product groups satisfying such a strong rigidity result. The following corollary enlarges this family of groups.

Corollary 6.14. Denote by \mathcal{M} the family of all wreath product groups $G = (\mathbb{Z}/2\mathbb{Z})^{(I)} \rtimes \Gamma$ where $\Gamma \curvearrowright I$ is any of the group actions $\Gamma \curvearrowright I$ on countable sets covered by Theorem 6.7 such that Γ has no finite normal subgroups and that $(\text{Stab } i) \cdot j$ is infinite for all $i, j \in I$ with $i \neq j$.

If $G_1, G_2 \in \mathcal{M}$, $t > 0$ and if $\pi : L(G_1) \rightarrow L(G_2)^t$ is a stable isomorphism between $L(G_1)$ and $L(G_2)$, then the groups G_1, G_2 are isomorphic through an isomorphism $\delta : G_1 \rightarrow G_2$, the amplification t equals 1 and there exist a unitary $w \in L(G_2)$ and a group morphism $\omega : G_1 \rightarrow S^1$ such that

$$\pi(u_g) = \omega(g) w v_{\delta(g)} w^* \quad \text{for all } g \in G_1.$$

Here, $(u_g)_{g \in G_1}$ and $(v_s)_{s \in G_2}$ denote the natural unitaries generating $L(G_1)$ and $L(G_2)$.

In particular, the automorphism group of $L(G)$ for $G \in \mathcal{M}$, is generated by $\text{Aut}(G)$, the group of characters $\text{Char } G$ and the inner automorphisms.

To prove Corollary 6.14, we use the following lemma, whose proof is contained in [PV06, Theorem 5.4]. Observe that Lemma 6.15 implies in particular that, whenever $\Gamma \curvearrowright I$ is such that $(\text{Stab } j) \cdot k$ is infinite for all $j \neq k$, the generalized Bernoulli actions $\Gamma \curvearrowright (X_0, \mu_0)^I$ and $\Gamma \curvearrowright (Y_0, \rho_0)^I$ are conjugate if and only if the base probability spaces are isomorphic. Such a statement is false in general for plain Bernoulli actions $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$, see [St75].

Lemma 6.15. Let $\Gamma_i \curvearrowright I_i$, $i = 1, 2$. Assume that for both actions, $(\text{Stab } j) \cdot k$ is infinite for all $j \neq k$. Consider the generalized Bernoulli actions

$$\Gamma_1 \curvearrowright (X, \mu) = (X_0, \mu_0)^{I_1} \quad \text{and} \quad \Gamma_2 \curvearrowright (Y, \rho) := (Y_0, \rho_0)^{I_2}.$$

If $\Delta : X \rightarrow Y$ is an isomorphism of probability spaces and $\delta : \Gamma_1 \rightarrow \Gamma_2$ an isomorphism of groups such that $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ for all $g \in \Gamma_1$ and almost all $x \in X$, there exists a bijection $\eta : I_1 \rightarrow I_2$ and there exist isomorphisms $\Delta_i : X_0 \rightarrow Y_0$ of the base probability spaces such that

- $\eta(g \cdot i) = \delta(g) \cdot \eta(i)$ and $\Delta_{g \cdot i} = \Delta_i$ for all $i \in I_1$, $g \in \Gamma_1$,
- $(\Delta(x))_{\eta(i)} = \Delta_i(x_i)$ for all $i \in I_1$ and almost all $x \in X$.

Proof of Corollary 6.14. Write $G_i = (\mathbb{Z}/2\mathbb{Z})^{(I_i)} \rtimes \Lambda_i$ and identify $L(G_i) = A_i \rtimes \Lambda_i$, where $A_i = L((\mathbb{Z}/2\mathbb{Z})^{(I_i)})$. Let $\pi : L(G_1) \rightarrow L(G_2)^t$ be a $*$ -isomorphism. By Theorem 6.7, we get that $t = 1$ and we get the existence of an isomorphism $\delta : G_1 \rightarrow G_2$, a character $\omega : \Lambda_1 \rightarrow S^1$ and a unitary $w \in L(G_2)$ such that

$$\pi(au_g) = \omega(g) w \alpha(a) v_{\delta(g)} w^*,$$

where $\alpha : A_1 \rightarrow A_2$ is a $*$ -isomorphism satisfying $\alpha(\sigma_g(a)) = \sigma_{\delta(g)}(a)$ for all $g \in \Lambda_1$ and $a \in A_1$.

Lemma 6.15 describes the form of $\pi|_{A_1}$. Since $L(\mathbb{Z}/2\mathbb{Z})$ has precisely two automorphisms, the identity and the multiplication with the non-trivial character on $\mathbb{Z}/2\mathbb{Z}$, we are done. \square

6.6 Counterexamples to W^* -superrigidity

There are many free ergodic p.m.p. actions $\Gamma \curvearrowright (X, \mu)$ with $\Gamma \in \mathcal{G}$ and such that $\Gamma \curvearrowright (X, \mu)$ is orbit equivalent with a large family of non-conjugate group actions. So, in these cases, the II_1 factor $L^\infty(X) \rtimes \Gamma$ has many group measure space decompositions (up to conjugacy of the actions), but all of them have the same Cartan subalgebra (up to unitary conjugacy).

First of all, by [Ga05, P_{ME}6], if Γ_i and Γ'_i admit orbit equivalent actions ($i = 1, 2$), then $\Gamma_1 * \Gamma_2$ and $\Gamma'_1 * \Gamma'_2$ also admit orbit equivalent actions.

Next, let Γ_1 be an arbitrary group and Γ_2, Γ'_2 infinite amenable groups. Denote $\Gamma = \Gamma_1 * \Gamma_2$ and $\Gamma' = \Gamma_1 * \Gamma'_2$. By [Bo09], the Bernoulli actions $\Gamma \curvearrowright (X_0, \mu_0)^\Gamma$ and $\Gamma' \curvearrowright (Y_0, \eta_0)^{\Gamma'}$ are orbit equivalent for all non-trivial base probability spaces (X_0, μ_0) and (Y_0, η_0) .

7 W^* -superrigidity for finite index bimodules

By definition, the action $\Gamma \curvearrowright (X, \mu)$ is stably W^* -superrigid if, for all free ergodic p.m.p. actions $\Lambda \curvearrowright (Y, \eta)$, stable isomorphism of $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$ implies, in a sense, stable isomorphism of the groups Γ, Λ and their respective actions.

For certain of the actions listed in Theorem 6.7, one can go even further and prove that, whenever \mathcal{H} is a finite index bimodule between $L^\infty(X) \rtimes \Gamma$ and $L^\infty(Y) \rtimes \Lambda$, the groups Γ, Λ and their actions are, in the following precise sense, virtually isomorphic, with \mathcal{H} being implemented by this virtual conjugacy and a finite dimensional unitary representation.

However, the existence of $g_1, \dots, g_n \in \Gamma$ with $\bigcap_{i=1}^n g_i \Sigma g_i^{-1}$ finite, does not provide sufficient absence of normality of $\Sigma < \Gamma$. Therefore, we make in the following theorem, a much stronger (and certainly non-optimal) assumption.

Theorem 7.1. *Let $\Gamma \curvearrowright (X, \mu)$ be any of the actions listed in Theorem 6.7 and assume that Γ admits an infinite index subgroup G such that $g \Sigma g^{-1} \cap \Sigma$ is finite for all $g \in \Gamma - G$. If $\Lambda \curvearrowright (Y, \eta)$ is an arbitrary free ergodic p.m.p. action and \mathcal{H} is a finite index $(L^\infty(X) \rtimes \Gamma) - (L^\infty(Y) \rtimes \Lambda)$ -bimodule, there exist*

- a finite index subgroup $\Gamma' < \Gamma$,
- a finite subgroup $H < \text{Aut}(X, \mu)$ such that $gHg^{-1} = H$ for all $g \in \Gamma'$,
- finite index subgroups $\Lambda'' < \Lambda' < \Lambda$ with $\Lambda \curvearrowright Y$ being induced from $\Lambda' \curvearrowright Y'$ for some $Y' \subset Y$,

such that $\Lambda'' \cong \frac{\Gamma'}{\Gamma' \cap H}$ and such that the action $\Lambda'' \curvearrowright Y'$ is conjugate with $\frac{\Gamma'}{\Gamma' \cap H} \curvearrowright \frac{X}{H}$, through an isomorphism $\Delta : \frac{X}{H} \rightarrow Y'$ of probability spaces and a group isomorphism $\delta : \frac{\Gamma'}{\Gamma' \cap H} \rightarrow \Lambda''$.

Moreover, if \mathcal{H} is irreducible, there exists an irreducible finite dimensional unitary representation $\pi : \Gamma' \rightarrow \mathcal{U}(\mathbb{C}^k)$ such that \mathcal{H} is the composition of

- the $(L^\infty(X) \rtimes \Gamma) - (L^\infty(X) \rtimes \Gamma')$ -bimodule on the Hilbert space $M_{1,k}(\mathbb{C}) \otimes L^2(M)$, where $M = L^\infty(X) \rtimes \Gamma$ and where

$$z \cdot \xi \cdot (au_g) = z \xi (\pi(g) \otimes au_g) \quad \text{for all } z \in M, a \in L^\infty(X), g \in \Gamma', \quad (7.1)$$

- the natural $(L^\infty(X) \rtimes \Gamma') - (L^\infty(\frac{X}{H}) \rtimes \frac{\Gamma'}{\Gamma' \cap H})$ -bimodule on the Hilbert space $L^2(M_0) p_{\Gamma' \cap H}$, where

$$M_0 = L^\infty(X) \rtimes \Gamma' \quad \text{and} \quad p_{\Gamma' \cap H} = |\Gamma' \cap H|^{-1} \sum_{g \in \Gamma' \cap H} u_g,$$

- the $(L^\infty(\frac{X}{H}) \rtimes \frac{\Gamma'}{\Gamma \cap H}) - (L^\infty(Y) \rtimes \Lambda)$ -bimodule on the Hilbert space $L^2(N)$, where $N = L^\infty(Y) \rtimes \Lambda$ and where

$$(au_g) \cdot \xi \cdot z = (a \circ \Delta^{-1})u_{\delta(g)}\xi z \quad \text{for all } a \in L^\infty(\frac{X}{H}), g \in \frac{\Gamma'}{\Gamma \cap H}, z \in N.$$

Example 7.2. Let Σ be an amenable group and $\Gamma = \Gamma_1 *_{\Sigma} \Gamma_2$. Assume that $\Gamma \curvearrowright I$ with $\Sigma \cdot i$ being infinite for every $i \in I$. If one of the following conditions is satisfied, the generalized Bernoulli action $\Gamma \curvearrowright (X_0, \mu_0)^I$ satisfies the conclusions of Theorem 7.1.

1. Let $\Sigma = \mathbb{Z}$ and embed Σ into $\Gamma_1 = \text{SL}(n, \mathbb{Z})$, $n \geq 3$, in the upper right corner. Assume that $\Sigma \triangleleft \Gamma_2$ is a proper normal subgroup. Then, Theorem 7.1 applies by putting $G = G_0 *_{\Sigma} \Gamma_2$, where G_0 consists of those matrices A with $A_{i1} = 0$ for all $2 \leq i \leq n$.
2. Let Σ be a common subgroup of $\Lambda_0, \Lambda_1, \Gamma_2$ and assume that Σ is normal in Γ_2 , $\Sigma \neq \Gamma_2$ and Λ_0 is non-amenable. Let Λ_2 be an arbitrary non-trivial group. Put $\Gamma_1 = \Lambda_0 \times (\Lambda_1 * \Lambda_2)$ and view $\Sigma < \Gamma_1$ diagonally. Theorem 7.1 applies by putting $G = (\Lambda_0 \times \Lambda_1) *_{\Sigma} \Gamma_2$. Assume that Λ_0 or $\Lambda_1 * \Lambda_2$ acts with infinite orbits on I .

In the proof of Theorem 7.1 we make use of the following lemma having some independent interest.

Lemma 7.3. Let $\Gamma \curvearrowright (X, \mu)$ and $\Lambda \curvearrowright (Y, \eta)$ be free p.m.p. actions. Assume that $\Gamma \curvearrowright (X, \mu)$ is cocycle superrigid with finite target groups and that all finite index subgroups of Γ act ergodically on (X, μ) . Let $\Delta : X \rightarrow Y$ be an m -to-1 measure preserving map satisfying $\Delta(g \cdot x) = \delta(g) \cdot x$ for some surjective group morphism $\delta : \Gamma \rightarrow \Lambda$.

Then there exists a group H with m elements and a free p.m.p. action $H \curvearrowright (X, \mu)$ such that, viewing both Γ and H as subgroups of $\text{Aut}(X, \mu)$, we have

- $gHg^{-1} = H$ for all $g \in \Gamma$,
- $\text{Ker } \delta = \Gamma \cap H$,
- Δ is the composition of the canonical quotient map $X \rightarrow \frac{X}{\frac{\Gamma}{\Gamma \cap H}}$ with a conjugacy of the actions $\frac{\Gamma}{\Gamma \cap H} \curvearrowright \frac{X}{\frac{\Gamma}{\Gamma \cap H}}$ and $\Lambda \curvearrowright Y$.

Proof. Partition X into disjoint measurable subsets X_1, \dots, X_m such that the restrictions $\Delta_i := \Delta|_{X_i}$ are isomorphisms of X_i onto Y scaling the measure by the factor $1/m$.

Put $\mathcal{J} := \{1, \dots, m\}$. For $g \in \Gamma$ and almost every $x \in X$, define the map $\omega(g, x) : \mathcal{J} \rightarrow \mathcal{J}$ such that

$$\omega(g, x)i = j \quad \text{if and only if} \quad g \cdot \Delta_i^{-1}(\Delta(x)) \in X_j.$$

So, by construction

$$g \cdot \Delta_i^{-1}(\Delta(x)) = \Delta_{\omega(g, x)i}^{-1}(\Delta(g \cdot x))$$

almost everywhere. It follows that $\omega(gh, x) = \omega(g, h \cdot x) \circ \omega(h, x)$ almost everywhere. Hence, $\omega(g, x)$ is a permutation of \mathcal{J} and ω is a 1-cocycle for the action $\Gamma \curvearrowright X$ with values in the permutation group S_m . Since $\Gamma \curvearrowright (X, \mu)$ is assumed to be cocycle superrigid with finite target groups, we find a measurable map $\varphi : X \rightarrow S_m$ and a group morphism $\eta : \Gamma \rightarrow S_m$ such that, writing

$$T_i : X \rightarrow X : T_i(x) = \Delta_{\varphi(x)i}^{-1}(\Delta(x)),$$

we have $g \cdot T_i(x) = T_{\eta(g)i}(g \cdot x)$ almost everywhere. By construction, every T_i is locally a m.p. isomorphism. Moreover, the range of T_i is globally invariant under the finite index subgroup $\text{Ker } \eta < \Gamma$. Hence, all T_i are m.p. automorphisms of (X, μ) .

Define the equivalence relation \mathcal{R} on X as $\mathcal{R} := \{(x, y) \in X \times X \mid \Delta(x) = \Delta(y)\}$. By construction, every equivalence class has m elements and the graph of every T_i belongs to \mathcal{R} . Finally, if $T_i(x) = T_j(x)$ for all x in a non-negligible subset \mathcal{U} , we can make \mathcal{U} smaller and assume moreover that $\varphi(x) = \sigma$ for all $x \in \mathcal{U}$. Hence, $\Delta_{\sigma i}^{-1}(\Delta(x)) = \Delta_{\sigma j}^{-1}(\Delta(x))$ for all $x \in \mathcal{U}$. So, $\sigma i = \sigma j$ and hence, $i = j$. We have shown that \mathcal{R} is the disjoint union of the graphs of the T_i , $i = 1, \dots, m$.

Define $H < \text{Aut}(X, \mu)$ as the group generated by T_1, \dots, T_m . By construction, all $h \in H$ commute with $\text{Ker } \eta$. Since $\text{Ker } \eta$ acts ergodically on (X, μ) , it follows that H acts freely on (X, μ) . Since the orbit equivalence relation of $H \curvearrowright X$ is contained in \mathcal{R} , we get that $|H| \leq m$. But H contains the distinct elements T_1, \dots, T_m , so that $H = \{T_1, \dots, T_m\}$.

It remains to prove that $\text{Ker } \delta = \Gamma \cap H$. Since $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ and Λ acts freely on Y , it follows that an element $g \in \Gamma$ belongs to $\text{Ker } \delta$ if and only if the graph of g belongs to \mathcal{R} . It follows that $|\text{Ker } \delta| \leq m$ and that $\Gamma \cap H \subset \text{Ker } \delta$. Conversely, if $g \in \text{Ker } \delta$, take i such that the graphs of g and T_i intersect non-negligibly. Since $\text{Ker } \delta$ is a finite normal subgroup of Γ , it follows that g commutes with a finite index subgroup of Γ . We have already seen that also T_i commutes with a finite index subgroup of Γ . It follows that g and T_i coincide almost everywhere, i.e. $g \in \Gamma \cap H$. \square

Proof of Theorem 7.1. As a preliminary step, note that Lemmas 3.1 and 3.2 remain valid when we identify $Q \rtimes \Lambda$ with a finite index subfactor of pMp . This generalization almost has the same proof and we explain this briefly for the case of Lemma 3.1. The deformation φ_n of M still allows to define the completely positive maps $\tilde{\varphi}_n$ on Λ , which in turn lead to a new deformation θ_n of $Q \rtimes \Lambda$. By Jones' tunnel construction and for the appropriate value of $s > 0$, we can view the amplification M^s as a finite index subfactor of $Q \rtimes \Lambda$. Hence, we find inside $Q \rtimes \Lambda$ a von Neumann subalgebra M_0 with the relative property (T) and without injective direct summand. So, we can finish the proof in the same way as for the original Lemma 3.1.

Write $A = L^\infty(X)$ and $M = A \rtimes \Gamma$. Put $B = L^\infty(Y)$ and $N = B \rtimes \Lambda$. We may assume that ${}_M \mathcal{H}_N$ is an irreducible finite index bimodule. Let $\gamma : N \rightarrow pM^m p$ be a finite index inclusion such that

$${}_M \mathcal{H}_N \cong {}_M((M_{1,m}(\mathbb{C}) \otimes L^2(M))p)_{\gamma(N)}.$$

View M^m as the crossed product $(M_m(\mathbb{C}) \otimes A) \rtimes \Gamma$ where Γ acts trivially on $M_m(\mathbb{C})$. As in Section 5.1, define for every $0 < \rho < 1$, the unital completely positive map m_ρ on M^m by $m_\rho(au_g) = \rho^{|g|}au_g$ for all $a \in M_m(\mathbb{C}) \otimes A$, $g \in \Gamma$.

In the situation where Γ_1 has a non-amenable subgroup with the relative property (T), apply Lemma 3.1 – as generalized in the first paragraph of the proof – to the completely positive maps m_ρ , $\rho \rightarrow 1$ and the von Neumann subalgebra $P = (A \rtimes \Sigma)^m$. In the situation where Γ_1 has two non-amenable commuting subgroups H_1, H_2 , we apply the generalized Lemma 3.2. Put $\varepsilon = (\text{tr} \otimes \tau)(p)/2072$. Combined with (5.1), we always find $0 < \rho < 1$ and a sequence $s_k \in \Lambda$ such that

- $\|\gamma(v_{s_k}) - m_\rho(\gamma(v_{s_k}))\|_2 \leq \varepsilon/2$ for all k .
- $\|E_P(x\gamma(v_{s_k})y)\|_2 \rightarrow 0$ for all $x, y \in M$.

By Lemma 5.7 we get a $0 < \rho < 1$ and a $\delta > 0$ such that $\tau(\gamma(w)^* m_\rho(\gamma(w))) \geq \delta$ for all $w \in \mathcal{U}(B)$. By Theorem 5.4 we have that $\gamma(B) \prec_M A \rtimes \Sigma$ or that $\mathcal{N}_{pM^m p}(\gamma(B))'' \prec_M A \rtimes \Gamma_i$ for some $i = 1, 2$. But

$\mathcal{N}_{pM^mp}(\gamma(B))$ contains $\gamma(N)$ which has finite index in pM^mp . Since $\Gamma_i < \Gamma$ has infinite index, this rules out the possibility that $\mathcal{N}_{pM^mp}(\gamma(B))'' \prec_M A \rtimes \Gamma_i$. So, we have shown that $\gamma(B) \prec_M A \rtimes \Sigma$.

We claim that $\gamma(B) \prec_M A$. Assume the contrary. Our assumption on G , together with the regularity of $B \subset N$, implies that $\gamma(N) \prec_M A \rtimes G$ (cf. [Va07, Lemma 4.2]). Since $\gamma(N) \subset pM^mp$ is of finite index, while $G < \Gamma$ has infinite index, this is a contradiction. So, we have shown that $\gamma(B) \prec_M A$.

In bimodule language, this means that \mathcal{H} admits a non-zero A - B -subbimodule \mathcal{K} which is finitely generated as a left A -module. Take a finite index inclusion $\psi : M \rightarrow qN^kq$ such that

$${}_M\mathcal{H}_N \cong {}_{\psi(M)}q(\mathbb{C}^k \otimes L^2(N))_N.$$

The existence of \mathcal{K} means that $B \prec_N \psi(A)$. Taking relative commutants (see [Va07, Lemma 3.5]), it follows that $\psi(A) \prec_N B$.

By Theorem 6.12 the action $\Gamma' \curvearrowright X$ is \mathcal{U}_{fin} -cocycle superrigid for every finite index subgroup $\Gamma' < \Gamma$. Therefore, we are in a situation where we can apply [Va07, Lemma 6.5] as well as the first part of the proof of [Va07, Theorem 6.4]. Denote by $D_k(\mathbb{C}) \subset M_k(\mathbb{C})$ the algebra of diagonal matrices. As a result, we get

- a finite index subgroup $\Gamma' < \Gamma$ and an irreducible projective representation $\pi : \Gamma' \rightarrow \mathcal{U}(\mathbb{C}^k)$ with obstruction 2-cocycle $\Omega : \Gamma' \times \Gamma' \rightarrow S^1$ given by $\pi(g)\pi(h) = \Omega(g, h)\pi(gh)$,
- a projection $q' \in D_k(\mathbb{C}) \otimes B$ and a finite index inclusion $\psi' : A \rtimes_{\Omega} \Gamma' \rightarrow q'N^kq'$ satisfying $\psi'(A) = (D_k(\mathbb{C}) \otimes B)q'$,

such that \mathcal{H} is the composition of the $(A \rtimes \Gamma) - (A \rtimes_{\Omega} \Gamma')$ -bimodule given by (7.1) and the $(A \rtimes_{\Omega} \Gamma') - N$ -bimodule given by

$$\psi'(A \rtimes_{\Omega} \Gamma')q'(\mathbb{C}^k \otimes L^2(N))_N.$$

Denote $\tilde{\Lambda} = \frac{\mathbb{Z}}{k\mathbb{Z}} \times \Lambda$ and $\tilde{Y} = \frac{\mathbb{Z}}{k\mathbb{Z}} \times Y$. Consider the natural action $\tilde{\Lambda} \curvearrowright \tilde{Y}$ and define $Z \subset \tilde{Y}$ such that $q' = \chi_Z$. Denote by $T : X \rightarrow Z$ the isomorphism of measure spaces determined by $\psi'(a) = a \circ T^{-1}$ for all $a \in A$. Normalize the measure η on Y in such a way that Z has measure 1 and hence, T is measure preserving.

By construction, $T(\Gamma' \cdot x) \subset \tilde{\Lambda} \cdot T(x)$ for almost all $x \in X$. So, the formula

$$T(g \cdot x) = \omega(g, x) \cdot T(x)$$

defines a 1-cocycle for the action $\Gamma' \curvearrowright X$ with values in $\tilde{\Lambda}$. By Theorem 6.12, $\Gamma' \curvearrowright X$ is \mathcal{U}_{fin} -cocycle superrigid. So, we find a measurable map $\varphi : X \rightarrow \tilde{\Lambda}$ and a group morphism $\delta : \Gamma' \rightarrow \tilde{\Lambda}$ such that, writing $\Delta(x) := \varphi(x) \cdot T(x)$, we have $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ almost everywhere. Since $\Gamma' \curvearrowright X$ is weakly mixing, we find $i \in \frac{\mathbb{Z}}{k\mathbb{Z}}$ such that $\Delta(x) \in \{i\} \times Y$ for almost all $x \in X$. Hence, $\delta(\Gamma') \subset \{1\} \times \Lambda$. From now on, we view Δ as a map from X to Y and δ as a group morphism from Γ' to Λ . Put $\Lambda'' := \delta(\Gamma')$.

By construction, Δ is locally a m.p. isomorphism, meaning that we can partition X into a sequence of measurable subsets X_n such that the restriction of Δ to each of the X_n is a m.p. isomorphism of X_n onto a subset of Y . Since $\Delta(g \cdot x) = \delta(g) \cdot \Delta(x)$ and since (X, μ) is a probability space, it follows that $\text{Ker } \delta$ is finite and that Δ is an m -to-1 quotient map onto $Y' \subset Y$ (cf. [Fu06, Theorem 1.8]). Observe that Y' is globally Λ'' -invariant. Partition X into subsets X_1, \dots, X_m of measure $1/m$ such that $\Delta_i := \Delta|_{X_i}$ is an isomorphism of X_i onto Y' .

Define the projection $p_{\text{Ker } \delta}$ in $A \rtimes \Gamma'$ by the formula

$$p_{\text{Ker } \delta} := |\text{Ker } \delta|^{-1} \sum_{g \in \text{Ker } \delta} u_g .$$

Put $B_1 := L^\infty(Y') = L^\infty(Y)q_1$, where $q_1 := \chi_{Y'}$. The pair Δ, δ yields a natural $(A \rtimes \Gamma') - (B_1 \rtimes \Lambda'')$ -bimodule structure on the Hilbert space $\mathcal{K} := L^2(A \rtimes \Gamma')p_{\text{Ker } \delta}$. We can also write this bimodule as

$$\gamma_{(A \rtimes \Gamma')}(\mathbb{C}^m \otimes L^2(B_1 \rtimes \Lambda''))_{B_1 \rtimes \Lambda''}$$

where $\gamma : A \rtimes \Gamma' \rightarrow M_m(\mathbb{C}) \otimes (B_1 \rtimes \Lambda'')$ is a finite index inclusion satisfying

$$\gamma(a) = \sum_{i=1}^m e_{ii} \otimes (a|_{X_i} \circ \Delta_i^{-1}) \quad \text{for all } a \in A .$$

Since $T(x) \in \tilde{\Lambda} \cdot \Delta(x)$ for almost all $x \in X$, we can take $W \in M_{m,k}(\mathbb{C}) \otimes N$ satisfying $WW^* = 1 \otimes q_1$, $W^*W = q'$ and $\gamma(a) = W\psi'(a)W^*$ for all $a \in A$. Replace ψ' by $W\psi'(\cdot)W$.

It follows that, for all $g \in \Gamma'$, $\psi'(u_g)\gamma(u_g)^*$ commutes with $\gamma(A) = \psi'(A) = D_m(\mathbb{C}) \otimes B_1$, which is maximal abelian in $M_m(\mathbb{C}) \otimes q_1 N q_1$. So, $\psi'(u_g) = \gamma(\omega_g u_g)$ where $\omega_g \in \mathcal{U}(A)$ satisfies

$$\omega_g \sigma_g(\omega_h) = \Omega(g, h) \omega_{gh} .$$

By Lemma 6.13 we find $w \in \mathcal{U}(A)$ and a map $\mu : \Gamma' \rightarrow S^1$ such that $\omega_g = \mu(g)w\sigma_g(w^*)$ for all $g \in \Gamma'$. Replacing $g \mapsto \pi(g)$ by $g \mapsto \overline{\mu(g)}\pi(g)$ and replacing ψ' by $(\text{Ad } \psi'(w)^*) \circ \psi'$, we may assume that π is an ordinary unitary representation (i.e. $\Omega = 1$) and that $\psi'(z) = \gamma(z)$ for all $z \in A \rtimes \Gamma'$. Since $\psi'(A \rtimes \Gamma')$ has finite index in $M_m(\mathbb{C}) \otimes q_1 N q_1$ and since $\gamma(A \rtimes \Gamma')$ is contained in $(B \rtimes \Lambda'')^m$, it follows that $\Lambda'' < \Lambda$ is a finite index subgroup.

If $s \in \Lambda$, then $\Delta^{-1}(s \cdot Y' \cap Y')$ is globally invariant under the finite index subgroup $\delta^{-1}(\Lambda'' \cap s \Lambda'' s^{-1})$ of Γ' and hence, must have measure 0 or 1. So, either $s \cdot Y' \cap Y'$ has measure zero, or $s \cdot Y'$ equals Y' up to measure zero. Define the subgroup $\Lambda' < \Lambda$ consisting of those $s \in \Lambda$ for which $s \cdot Y'$ equals Y' up to measure zero. By construction, $\Lambda'' < \Lambda' < \Lambda$ and $\Lambda \curvearrowright Y$ is induced from $\Lambda' \curvearrowright Y'$.

The theorem now follows by applying Lemma 7.3 to the actions $\Gamma' \curvearrowright (X, \mu)$ and $\Lambda'' \curvearrowright Y'$. \square

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